Rhizomatic Thinking and Voting Equilibria in Large Multi-Candidate Elections under Plurality Rule*

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Abstract

We consider a model of strategic voting behavior in large multi-candidate elections under Plurality Rule where we allow for the possibility of rhizomatic thinking (Bravo-Furtado & Côrte-Real [4]). Our rhizomatic assumption states that each agent may, to various degrees, condition her optimal response on an exogenous belief she possesses over the proportion of like-minded others that will take the same action as she does. In our pivotal-agent game, we therefore relax self-goal choice - and rhizomatic beliefs will induce agents to perceive that they may be pivotal with a non-vanishing probability.

We modify the trinomial pivotal-voter model of Palfrey [21] and solve for asymptotic equilibria using appropriate techniques provided by large deviations theory, given the distributions of preferences and beliefs. We show existence and possible uniqueness of equilibria in this setting. We then conclude that our model may help select equilibria, adding predictive power to standard game-theoretic settings. We also find that Duverger's Law may be violated in equilibrium and, as an application, we suggest rhizomatic thinking can help provide a compelling rationale for the non-Duvergerian outcome of 1992 US presidential election. Correlations between rhizomatic beliefs and preferences explain the different equilibrium outcomes and our model therefore calls attention to the role of partisanship and group identity in plurality election outcomes.

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1 Introduction

Large elections have an obvious importance in democratic life and the design of optimal voting rules must take into account its impact on voting behavior. Standard notions of individual rationality may however fail to fully account for observed behavior by voters in large elections. Our main purpose is to research the impact that the concept of Rhizomatic Thinking (RT) introduced by Bravo-Furtado and Côrte-Real [4] can have in this setting.

In Biology, "rhizome" is the term for a usually underground horizontal stem of a plant that often sends out roots and shoots from its nodes. Deleuze and Guatarri [7] generalize the concept of a rhizome as an acentered and non-hierarchical system made up of possibly heterogeneous connections. We borrowed this notion of a rhizome to introduce the concept of RT. Our rhizomatic behavior assumption states that each agent i may, to various degrees, condition her optimal response on an exogenous rhizomatic belief (RB), $q_i \in [0,1]$, she possesses over the proportion q_i of "like-minded" others that will take the same action as she does. Each agent therefore interprets her own decision as a diagnosis of the decisions that this proportion of "like-minded" others will take. The class of "like-minded" others is game specific, and in most cases it corresponds to the class of agents who share that agent's preferences. If each agent's RB is taken to be zero, we recover the standard game-theoretic framework and our assumption can therefore be seen as a generalization of the classical assumption.

We have argued that this may be a unifying assumption on rationality that allows for a general application in games (Bravo-Furtado and Côrte-Real [4]), and that may be particularly useful in collective action settings. If this is the case, RT may provide a strong additional rationale for instrumental behavior by agents in pivotal-agent games (PAG), especially when the number of players is very large. In such cases, game theory prescribes that purely individually rational agents condition their actions on the event that they are pivotal (an event with an infinitesimal likelihood). It is questionable to what an extent such a rationale may plausibly explain relevant instrumental motivations for voluntary participation and/or strategic behavior in large elections when compared to expressive motivations, for example. However, rhizomatic agents may no longer perceive their pivot probability as infinitesimally small, which may therefore enhance the plausibility of such instrumental behavior. RT may therefore add explanatory and predictive power and produce results that differ substantially from those of standard game-theoretic models. Our purpose is to study the effects of introducing RT in PAG, namely by modeling voting behavior in large multi-candidate elections under Plurality Rule (PR).

We consider the population of voters who actually do vote or, equivalently, we do not allow for abstention. It is important to note that RB not only provide agents with additional instrumental motivation to vote strategically but also to participate in elections. However, whereas strategic decisions regarding candidate choice cannot be justified by expressive motivations, it is more difficult to isolate the effects of expressive motivations and rhizomatic thinking with respect to participation. Although it is debatable whether the two decisions are separable, we therefore choose to focus our analysis on candidate choice.

Our model is based on the trinomial model presented by Palfrey [21]. The key difference is that we now endow each agent with a RB besides the standard linear ordering over candidates, where both preferences and the distributions of RB are common-knowledge. The multinomial model turns out to be most appropriate in order to incorporate RT as well as any alternative behavioral assumptions. Our model can be seen as an extension of Palfrey's, which results from setting all RB equal to zero. However, our main contribution may be the particular nature of this extension: we enrich the model by adding diversity from a behavioral viewpoint, rather than by adding informational uncertainty. The latter approach has widely been undertaken in the literature (see Myatt [20] for a recent example). It is also worth remarking that solving our model analytically for asymptotic equilibria requires different techniques from those used by Palfrey [21]. Fortunately, large deviations theory provides fully adequate and easier-to-handle tools to this effect, which are required in order to obtain accurate approximations for the tails of the multinomial distribution.

Palfrey's main result establishes Duverger's Law as a regularity: only two-party equilibria arise as possible outcomes, since voters who prefer the least voted-for candidate respond by voting for their second-ranked candidate in any equilibrium, whereas all other voters vote for their top-ranked candidate. In fact, as Laslier [15] points out. any classical pivotal-voter model in which the distribution of preferences is common-knowledge – such as the Poisson-Myerson model with population uncertainty -, will produce the same result under PR. In our model, however, the rhizomatic best responses of voters who prefer the least voted for candidate will generically depend on their RB in any equilibrium (whereas all other voters still vote for their top-ranked candidate). Such best responses are characterized by cutoff levels of the RB for each type, which are a function of the distributions of preferences and RB, depending also on the equilibrium probabilities. Voters who prefer the least voted-for candidate and whose RB is above the threshold level for her type[1] will now vote for their most preferred candidate. This is the first main result of the paper and allows us to characterize the equilibria of the voting game.

We then show existence of equilibria and proceed to analyze examples that allow for predictions that differ from Palfrey's. On one hand, uniqueness of equilibrium is possible and unique orderings can be achieved even if there is no unique equilibrium: equilibrium selection will in general result from RT, refining the standard plethora of equilibria in large elections and endowing the theory with a stronger predictive power.

We also conclude that if there is one preference ordering that is shared by at least half of the voters then that alternative always wins provided agents are rhizomatic "enough"; in this particular case, we eliminate couterintuitive Duvergerian equilibria in which such an alternative would come in last. Moreover, three-party equilibria may occur and Duverger's Law may fail to hold.

[1] If any exist, and whenever this threshold is below 1.

This is an important consequence of our model and it may help explain observed non-Duvergerian outcomes in plurality elections. This may help explain why Duverger's Law is not a regularity in elections in India (where in fact it hardly ever holds) and in constituency-level elections in the UK, to cite the most well-known exceptions (Myatt [20]). It can also provide a rationale for other examples such as the 1992 US presidential election where Ross Perot received a significant proportion of the popular vote in almost all states. We discuss in detail the three-candidate outcome of the Bush-Clinton-Perot presidential race and show that RT and correlations between preferences and beliefs can add insight to this particular case.

In general, vote manipulation is (weakly) decreasing in RB but the driving forces that may bring about different equilibrium outcomes are the distributions of preferences and rhizomatic beliefs, including the respective correlations. We therefore suggest that besides preference formation, there is another potential field of interest for political or party strategists: campaigns may influence the formation of rhizomatic beliefs, whether by creating feelings of group-identity among partisans or by appealing to strategic voting from supporters for other parties.

The paper proceeds as follows:

Section 2 reviews the literature. Due to the multi-disciplinary scope of our analysis, we split it twofold. We first examine the determinants of voting behavior and argue that RT can capture alternative rationales proposed in the literature to explain strategic voting behavior in PAG. We then proceed to focus on the regularity of Duverger's Law. We then.review Bravo-Furtado & Côrte-Real [4] and argue in detail why we find RT to be an appropriate and unifying notion of rationality for modeling behavior in games and specifically in collective action settings such as PAG.

In section 3 we present and solve our model asymptotically. We specify the best responses for all agents and characterize the asymptotic equilibria resorting to technical results from large deviations theory (results that we summarize in Appendix A). We proceed to discuss the limitations of the model and the main characteristics of our results, the assumptions, and possible extensions and generalizations. We then argue that simple discrete distributions of RB can be used to capture fairly general distributions with no loss of generality and with considerable gains in terms of insight. This allows us to understand equilibrium selection and the effects of correlations between RB and preferences.

Section 4 studies illustrative examples that allow us to understand the mechanisms that drive equilibrium selection and to achieve possible policy implications.

Section 5 concludes and summarizes the main findings.

2 Literature Review

Standard game theory assumes - and prescribes - individual rationality as the sole determinant of each agent's behavior. According to Sen [30], individual rationality is characterized by three independent and complementary assumptions: self-centered welfare ("a person's welfare depends only on his or her own consumption"), self-welfare goal ("a person's only goal is to maximize his or her own welfare" and not "the welfare of others"), and self-goal choice ("each act of choice of a person is guided immediately by the pursuit of one's own goal" and "in particular, is not restrained by the recognition of other people's pursuit of their goals"). Several authors have long criticized this assumption, namely in collective action settings, on both positive and normative counts (see Sen [27], [29], [30]). However, no alternative unifying concept of rationality has so far been formalized in order to address this criticism.

A related line of criticism with respect to the concept of individual rationality, pointed out by Sen [28] and Gilboa et al. [12], is its applicability to large games such as elections. In the past two decades, studies of large elections under game-theoretic settings (such as Palfrey's [21]) have provided strong theoretical support in favor of Duverger's Law : in a three-candidate election, there would only be two-party equilibria. However, empirically this is not the only observed outcome – constituency-level elections on the UK and India are well-known exceptions. One theoretical exception to the prediction of Duvergerian outcomes is that of Myatt [20]. Myatt's approach is based on signals and informative voting and his three-party equilibrium predictions rely solely on voters' uncertainty with respect to the identity of the leading defiant candidate. Myatt's contextualization therefore requires that polls, or alternative informational mechanisms, are sufficiently uninformative about the relevant priors to all voters - and other settings would still lead to Duvergerian outcomes.

Standard notions of individual rationality applied to large elections seem to be unable to explain non-Duvergerian outcomes but also fail to explain why only certain focal equilibria emerge in practice - and are unable to explain turnout as well. Recent voting literature has started to question "full" individual rationality (see Dhillon and Peralta [9] for a survey of the literature on turnout), as well as the connection between preference and choice (see, for instance, Côrte-Real [6]). Over the past few years, political scientists (Felsenthal et al. [11], Felsenthal [10], McKelvey & Ordeshook [16] and [18], Rietz [23], Scotto & LaFone [25]) have also shown an increasing interest in the effects of strategic voting behavior on coordination and political representation in multi-candidate elections, for which there is abundant empirical and experimental evidence - but that still lacks an adequate theoretical model.

Other scientific branches suggest alternative rationales for collective action. Agents may believe that their voting decision induces others to vote likewise (magical thinking). Alternatively, agents may merely believe that their own actions are a diagnosis of collective behavior, that is, they may read their own vote as a sign that many like-minded others will vote too, and perhaps with the same voting pattern ("illusory correlation" - beliefs that inaccurately suppose a relationship between a certain type of action and an effect).

Goldberg et al. [13] suggest individual agents may follow a "what if everyone acted that way" type of reasoning, referred to in the Cognitive Sciences literature as the "voter's illusion" or, equivalently, as the mixture of "symmetry" and the "illusion of control".

With the paper by Brams [3] on Newcomb's problem and the prisoner's dilemma, and the influential work of Quatronne & Tversky [22], collective reasoning - namely the "voter's illusion" - has emerged as a likely determinant for strategic voting behavior and coordinated outcomes. It is the one for which experimental evidence is most consistent (Morris et al. [19], Acevedo & Krueger [1]) and provides a rationale for strategic behavior with less demanding computational requirements. In its weakest version, agents consider their own behavior as a diagnosis of the behavior of like-minded others (agents with the same preferences or in the same "group"). However, instead of considering this feature as a departure from rationality, some authors argue that the correct rationality concept has to be thought of as an integrating part of the context that it is used in (Acevedo & Krueger [1]).

Also, models that allow for explicit individual bounded-rationality seem to be of little use in the context of large collective action games.

One such example is the hybrid model by Camerer, Palfrey & Rogers [5], that captures both cognitive hierarchy (CH) and quantal response (QR) equilibrium models. However, CH models are not computable for large games, and QR models assume that players have decreased payoff responsiveness. The correct direction for any strategic pivotal model of voting capable of increasing predictive power seems to demand an assumption that translates into the model by making agents act as if they were more often pivotal than what they really are.

In Bravo-Furtado & Côrte-Real [4], we argued extensively why RT may be an appropriate concept to model collective action, particularly in large games, reconciling individual and more collective notions of rationality.

An agent will think rhizomatically if she believes that a proportion of like-minded others will take the same action as she does. The agent therefore perceives a connection (that is actually "inexistent") among herself and like-minded others. By allowing for RT, we are therefore allowing for the mental mechanism to be such that the agent believes that she belongs to a network with no actual links, and such that she acts in accordance with that belief. We thus add a new variable to the classical game-theoretic setting: an exogenous belief that is an idiosyncratic feature of the agent i.e. a feature of an agent's "privateness" (Sen [30]).

Anderson [2], in his analysis of nationalism, states that «A nation is "an imagined political community". It is imagined because the members of even the smallest nation will never know most of their fellow-members, meet them, or even hear of them, yet in the minds of each lives the image of their communion. (...) In fact, all communities larger than primordial villages of face-to-face contact (and perhaps even these) are imagined.». The idea of an imagined community is one of the components of RT; the second important component is the restriction on an agent's action imposed by this "imagined community".

A person who is reading a book while waiting for a medical appointment and who decides to close the book at a given moment may perceive that several other people around the world performed that same action at the same time, and yet perceives no connection among those actions. On the other hand, in a collective action context such as voting, the perception of simultaneity may also involve the perception of connection among actions. This example suggests that mental processes matter and are liable to depend on specific contexts, like Rubinstein and Salant [24] point out. RT seems capable of encompassing this diversity and context-dependence in a general and unifying way.

We should stress that RT relaxes only the self-goal choice assumption, even though self-interest may well continue to drive agents' behavior – and it encompasses individual rationality as well as a special case. Moreover, RT can also capture the ideas of collective reasoning, mutual interdependence, group identity (and "impure altruism"), as well as cognitive biases such as the belief in personal relevance, the voter's illusion and the illusion of control[2].

The tension between individual and collective notions of rationality becomes particularly prominent in collective action settings. In order to address several open questions regarding voting behavior in a large election (where group identity may indeed play a significant role), namely to examine the validity of Duverger's Law and the selection of focal equilibria, we therefore introduce the concept of RT.to model a large voting game under PR.

3 The Model

We first define the fundamental concepts.

Consider a normal form game with N players. Let S_i denote the space of pure strategies and let $\Sigma_i = \Delta(S_i)$ be the space of mixed strategies. Let $N_i \subseteq N$ be the game-specific set of agent i's like-minded agents. Let $q_i \in Q_i \subseteq [0, 1]$ be the rhizomatic type of agent i, that is, the proportion of agents in N_i that agent i believes will take the same action as she does. Let λ_i stand for all components of i's type other than i's rhizomatic belief. Player i's type can then be denoted by $t_i = (\lambda_i, q_i)$. We assume that t_i is known only to player i. Let F denote a (common-knowledge) cumulative distribution function (cdf) of types and let F_i denote the marginal cdf for player i.

Definition 1 *A rhizomatic strategy for agent i is a map $p_i : T_i = \Lambda_i \times Q_i \to \Sigma_i$.*

Let $u_i(p_i, p_{-i}, t_i \,|t_i\,)$ denote player i's expected payoff associated with playing p_i when all others are playing p_{-i}, given that her type is t_i. As usual, $-i$ denotes $j = 1, ..., N$ with $j \neq i$.

Note that the choice of terminology is meant only to emphasize the exogeneity of beliefs q_i, which are now part of an agent's type: in all other regards, the definitions are equivalent to those in Bayesian Nash settings.

Definition 2 *A rhizomatic strategy p_i is a Rhizomatic Best Response (RBR) to p_{-i} if for all p_i' and for all t_i'*

[2] In any case, mental processes are very different under each assumption. The "voter's illusion" cognitive bias implies that agents actually perceive an "illusion of control": their decision is perceived as influencing others to act likewise, whereas RT is a horizontal notion.

$$E_{t_{-i}}(u_i(p_i(t_i'), p_{-i}(t_{-i}), t_i')\,|t_i')) \geq E_{t_{-i}}(u_i(p_i', p_{-i}(t_{-i}), t_i')\,|t_i')).$$

Definition 3 *A Rhizomatic Nash Equilibrium (RNE) is a vector of rhizomatic strategies* $(p_1, ..., p_N)$ *such that,* $\forall i, p_i$ *is a RBR to* p_{-i}*.*

Definitions 2 and 3 characterize the equivalent of Bayesian Nash Best Response and Bayesian Nash Equilibrium under a common-knowledge distribution of priors with private realizations of types.

We can now present the model.

There are three candidates, A, B and C, and polity with a large number N of voters.

Each voter i has a *strict preference ordering* over the three candidates. A voter's preference for each candidate is represented by a Von Neumann-Morgenstern utility number, normalized so that voter i receives a utility of 1 if her preferred candidate wins, a utility of 0 if her least preferred candidate wins, and a utility of $\lambda_i \in (0, 1)$ if the winner is her middle-ranked candidate.

Each voter also possesses an exogenous *rhizomatic belief,* $q_i \in [0, 1]$, as to the proportion of like-minded others she believes will always choose the same action as she does.

Let a $jk-$*voter* denote a voter whose preferred candidate is j and whose second-preferred candidate is k. Let l_{jk} represent the *average proportion of* $jk-$*voters* in the electorate. Hence, $l_{AB}+l_{AC}+l_{BA}+l_{BC}+l_{CA}+l_{CB} = 1$. The relevant rhizomatic class of "like-minded" others for a $jk-$voter is the class of other $l_{jk}N - 1$ $jk-$voters plus herself[3].

Let a $jk-$*type* denote a $jk-$voter whose utility if k wins is λ_i, and whose rhizomatic belief is q_i. Let $F_{jk}(.,.)$ represent the cumulative distribution function (cdf) of $jk-$types.

We assume that l_{jk} and $F_{jk}(.,.)$ are common knowledge, for all $j, k \in \{A, B, C\} : j \neq k$, and that each voter's type is drawn independently from the probability distribution $P = (l_{AB}, ..., l_{CB}, F_{AB}, ..., F_{CB})$.

We also assume that no voters abstain and introduce the following additional assumptions:

Assumption 1 $l_{jk} > 0$, for all $j, k \in \{A, B, C\} : j \neq k$.

Assumption 2 $F_{jk}(.,.)$ is twice continuously differentiable for $(\lambda_i, q_i) \in (0,1)\times(0, 1)$ and $\frac{\partial}{\partial\lambda}F_{jk}(\lambda, q) > 0, \forall(\lambda, q) \in (0, 1)\times(0, 1)$, for all $j, k \in \{A, B, C\}$: $j \neq k$. Furthermore, $F_{jk}(0, q_i) = 0$, $F_{jk}(1, q_i) = G_{jk}(q_i)$ and $G_{jk}(0) = q_{jk} \in [0, 1]$.

Assumption 1 states that all possible preference rankings occur: this is a minor technical assumption and its only purpose is allowing to state our results for all possible polities. It can, however, be easily relaxed in a straightforward manner without affecting the results.

[3]Note that a voter only knows the average number of "like-minded" others; this uncertainty vanishes as $N \to \infty$.

Assumption 2 rules out mass points in λ_i and in $q_i \in (0,1)$. It ensures it is a zero-measure event for two voters to have the same preference ordering and the same intensity of preference λ_i (this can also be straightforwardly relaxed in our asymptotic setting). However, we allow for mass points in $q \in \{0,1\}$. On one hand, we would like to allow for a proportion of voters not to be rhizomatic at all; on the other hand, this allows us to recover the standard framework (if this proportion is equal to 1). The possibility of mass points at $q = 1$ will not affect our results and will allow us to analyze examples of discrete distributions as well.

The voting game is standard. Under PR, each voter simultaneously chooses one candidate to vote for and the candidate with the highest number of votes will win. Following Palfrey [21], and for simplicity, ties will be broken alphabetically (e.g. A beats B in a tiebreaker). This assumption will not affect our results since ties are zero-measure events when $N \to \infty$.

We will look for *(pure strategy) symmetric Rhizomatic Nash Equilibria*, i. e. Bayes-Nash equilibria conditioned on a voter's type (which includes her rhizomatic belief), in which no voter chooses a weakly dominated strategy. A *(pure) strategy for agent i* under PR is therefore a measurable function

$$\sigma_i^{PR} : \{AB, AC, BA, BC, CA, CB\} \times (0,1) \times [0,1] \to \{A, B, C\} \tag{1}$$

In a pure-strategy symmetric Rhizomatic Nash Equilibrium $\sigma_i^{PR} = \sigma_{PR}$ for all i.

3.1 Plurality Rule (PR)

Let $[X]$ denote the highest integer smaller than or equal to X. A $jk-$type perceives the number of agents that will take the same action as she does (including herself) to be

$$M_{jk}^N = 1 + [q_i(Nl_{jk} - 1)]^{4}.$$

Let $O_{jk}^N = N - M_{jk}^N = N - 1 - [q_i(Nl_{jk} - 1)]$ be the set of agents that i does not perceive as "linked" to her. Let π_j denote the probability a randomly selected voter out of O_{jk}^N votes for j, such that $\pi_A + \pi_B + \pi_C = 1, \pi_j \geq 0$. For any σ_{PR} and (π_A, π_B, π_C), we can characterize a Rhizomatic Best Response for any voter i. We do so, without loss of generality, for an AB-type.

Let:

p_{AB}^N = probability that voting for A yields A, but voting for B yields B.
p_{AC}^N = probability that voting for A yields A, but voting for B yields C.
p_{CB}^N = probability that voting for A yields C, but voting for B yields B.

[4] We assume that, whenever $1+q_i(Nl_{jk}-1)$ is not an integer, i perceives $\left[1 + q_i(Nl_{jk} - 1)\right]$ as the number of agents that take the same action as she does.

Lemma 4 *(adapted from Palfrey [21]): If voter i is an AB-type, and if the remaining O_{AB}^N voters that i does not perceive as "linked" to her use σ_{PR}, generating probabilities (π_A, π_B, π_C), then i's Rhizomatic Best Response is:*

i) Vote for A if $p_{AB}^N(1-\lambda_i) + p_{AC}^N > p_{CB}^N\lambda_i$.

ii) Vote for A if (i) holds with equality (voting for B would be a dominated strategy).

iii) Vote for B if $p_{AB}^N(1-\lambda_i) + p_{AC}^N < p_{CB}^N\lambda_i$.

Note that, whenever $M_{AB}^N \geq O_{AB}^N \Leftrightarrow M_{AB}^N \geq \frac{N}{2}$, an AB type always votes for A.

In all other non-trivial cases, that is, whenever $M_{AB}^N < O_{AB}^N \Leftrightarrow M_{AB}^N < \frac{N}{2}$, the probabilities p_{AB}^N, p_{AC}^N and p_{CB}^N result from the trinomial distribution with parameters O_{AB}^N and $\pi_j, j = A, B, C$. We derive below these probabilities for an AB type, without loss of generality. For ease of notation, we will henceforth omit the subscript i and also the superscript N and the subscript AB from O_{AB}^N and M_{AB}^N. Let a be the number of other O voters who vote for A, b who vote for B and c who vote for C[5]. Hence,

$$a + b + c = O$$

Note that $M + a \geq b \wedge M + a \geq c$ are the conditions required for A winning when i votes for A, and $M + b < c \wedge a < c$ are the conditions for a victory of C when i votes for B. .Thus, p_{AC}^N is the probability that out of O other voters, a vote for A, b vote for B, c vote for C, and $M + a \geq b \wedge M + a \geq c \wedge M + b < c \wedge a < c \wedge a, b, c \geq 0$ i.e. $p_{AC}^N = \Pr(c > a \geq c - M > b \geq 0)$.

Let the number of votes for A, B and C respectively be k, $O - 2k - l$, and $k + l$. From the above conditions, we get:

$$\begin{aligned} c &> a \Leftrightarrow l > 0 \\ a &\geq c - M \Leftrightarrow l \leq M \\ c - M &> b \Leftrightarrow k > \frac{N-2l}{3} \\ \text{and finally, } b &\geq 0 \Leftrightarrow k \leq \frac{O-l}{2} \end{aligned}$$

which yields

$$p_{AC}^N = \sum_{l=1}^{M} \sum_{k=\left[\frac{N-2l}{3}\right]+1}^{\left[\frac{O-l}{2}\right]} \binom{O}{k}\binom{O-k}{k+l} \boldsymbol{\pi}_A^k \boldsymbol{\pi}_B^{O-2k-l} \boldsymbol{\pi}_C^{k+l}$$

In a similar way, p_{CB}^N is the probability that out of O other voters, a vote for A, b vote for B, c vote for C, and $M + a < c \wedge b < c \wedge M + b > a \wedge M + b \geq c \wedge a, b, c \geq 0$ i.e. $p_{CB}^N = \Pr(c > b \geq c - M > a \geq 0)$.

[5] We are also omitting the subscript N for a, b and c.

Let the number of votes for B be k, for C, $k+l$, and for A, $O-2k-l$. Then,

$$p_{CB}^{N}=\sum_{l=1}^{M}\sum_{k=\left[\frac{N-2l}{3}\right]+1}^{\left[\frac{O-l}{2}\right]}\binom{O}{k}\binom{O-k}{k+l}\pi_A^{O-2k-l}\pi_B^{k}\pi_C^{k+l}$$

Also, p_{AB}^{N} is the probability that out of O other voters, a vote for A, b vote for B, c vote for C, and $M+a\geq b\wedge M+a\geq c\wedge M+b>a\wedge M+b\geq c\wedge a,b,c\geq 0$.

In this case, it helps splitting the above conditions into two mutually exclusive cases: $a\geq b$ and $a<b$.

Thus, $p_{AB}^{N}=\Pr(M+b>a\wedge M+b\geq c\wedge b\geq 0\wedge c\geq 0\wedge a\geq b)+\Pr((M+a\geq c\wedge M+a\geq b\wedge a\geq 0\wedge c\geq 0\wedge a<b)=$

$$\begin{aligned}p_{AB}^{N} &= \sum_{l=0}^{M-1}\sum_{k=\max\{0,R\}}^{\left[\frac{O-l}{2}\right]}\binom{O}{k}\binom{O-k}{k+l}\pi_A^{k+l}\pi_B^{k}\pi_C^{O-2k-l}+\\ &\quad+\sum_{l=1}^{M}\sum_{k=\max\{0,R\}}^{\left[\frac{O-l}{2}\right]}\binom{O}{k}\binom{O-k}{k+l}\pi_A^{k}\pi_B^{k+l}\pi_C^{O-2k-l}\end{aligned}$$

where $R=\begin{cases}\left[\frac{O-M-l}{3}\right]+1, \text{ if } \frac{O-M-l}{3}\notin\mathbb{N}\\ \frac{O-M-l}{3}, \text{ if } \frac{O-M-l}{3}\in\mathbb{N}\end{cases}$.

Note that, as it ought to be the case, the expressions for the probabilities above reduce to those in Palfrey [21] when $q=0$[6].

From Lemma 1, if $p_{jk}^{N}>0\vee p_{lk}^{N}>0$, inequality (i) and equality (ii) yield

$$\lambda_{jk}\leq\frac{p_{jk}^{N}+p_{jl}^{N}}{p_{jk}^{N}+p_{lk}^{N}}, j,k,l=A,B,C\wedge j\neq k\neq l \tag{2}$$

As in Palfrey [21], strategic voting requires λ to be large enough and that the voter is more likely to be pivotal between her second and least preferred candidate than between her first and last.

In order to fully characterize a RNE, we must impose that the values of the probabilities π_j, $j=A,B,C$, are in fact confirmed in equilibrium. An equilibrium is therefore characterized by a set of six cutpoints, λ_{jk}^{N}, $j,k\in\{A,B,C\}:j\neq k$, each of which must either be 1 or satisfy condition 2 with equality. Since the cutpoints depend on rhizomatic beliefs in a way to be specified ahead, we delay the presentation of the asymptotic equilibrium conditions that confirm the equilibrium probabilities until the necessary results to this effect are established.

[6] The lower and upper bounds of the summations in Palfrey (1989) contain a few obvious typos

3.2 Asymptotic Equilibria

Since we are considering very large electorates ($N \to \infty$), we can establish the asymptotic properties of the probabilities derived above for a $jk-$type with $q > 0$ (for $q = 0$, the results coincide with Palfrey's) and therefore determine the best responses for each type of voter. The following two Propositions are crucial for our results. With no loss of generality, we state them for an $AB-$type.

Let $x \equiv x_{AB} = \frac{ql_{AB}}{1-ql_{AB}}$. Let Δ denote the two-dimensional simplex. Let:

$$
\begin{aligned}
E_{AC} &= \left\{(\pi_A, \pi_C) \in \Delta : \pi_C > \pi_A \wedge \pi_C \leq x + \pi_A \wedge \pi_C > \frac{1+x}{2} - \frac{\pi_A}{2}\right\} \\
E_{CB} &= \left\{(\pi_A, \pi_C) \in \Delta : \pi_C > \frac{1}{2} - \frac{\pi_A}{2} \wedge \pi_C \leq \frac{1+x}{2} - \frac{\pi_A}{2} \wedge \pi_C > x + \pi_A\right\} \\
E_{AB} &= \left\{\begin{array}{c} (\pi_A, \pi_C) \in \Delta : (\pi_C \geq 1 - 2\pi_A \wedge \pi_C < \frac{1+x}{2} - \frac{\pi_A}{2} \wedge \pi_C \leq 1 + x - 2\pi_A \\ \vee \pi_C < 1 - 2\pi_A \wedge \pi_C \leq x + \pi_A \wedge \pi_C \geq 1 - x - 2\pi_A) \end{array}\right\}
\end{aligned}
$$

Let $closure(E_{jk})$ denotes its closure and let $ext(E_{jk})$ denotes the exterior of E_{jk}. Then $ext(E_{jk}) = \Delta \setminus closure(E_{jk})$.

Proposition 5 *Let Assumptions* $1 - 2$ *hold. Let voter* i *be an* AB *type with* $q > 0$. *Then,*

$$
i)\ p^N_{AC} \underset{N\to\infty}{\to} \begin{cases} 1, \text{ if } (\pi_A, \pi_C) \in E_{AC} \\ 0, \text{ if } (\pi_A, \pi_C) \in ext(E_{AC}) \end{cases}
$$

$$
ii)\ p^N_{CB} \underset{N\to\infty}{\to} \begin{cases} 1, \text{ if } (\pi_A, \pi_C) \in E_{CB} \\ 0, \text{ if } (\pi_A, \pi_C) \in ext(E_{CB}) \end{cases}
$$

$$
iii)\ p^N_{AB} \underset{N\to\infty}{\to} \begin{cases} 1, \text{ if } (\pi_A, \pi_C) \in E_{AB} \\ 0, \text{ if } (\pi_A, \pi_C) \in ext(E_{AB}) \end{cases}
$$

Proof. The proof results straightforwardly from the strong Law of Large Numbers (LLN) and from the inequalities involving $a = a_N, b = b_N$ and $c = c_N$, that define each of the probabilities above.

If $a_N, c_N \sim trinomial\,(O, \pi_A, \pi_C)$, then the sequence (a_N, c_N) satisfies the strong LLN:

$$
\frac{a_N}{O} \underset{N\to\infty}{\overset{a.s.}{\to}} \pi_A, \frac{c_N}{O} \underset{N\to\infty}{\overset{a.s.}{\to}} \pi_C
$$

so that

$$
P\left(\lim_{N\to\infty} \frac{j_N}{O} - \pi_j = 0\right) = 1,\ j = a, b, c
$$

i) From $c > a \geq c - M > b \geq 0$, replacing $b = O - a - c$, dividing all inequalities by O and taking limits as $N \to \infty$, we obtain, from the strong LLN that the result must hold whenever

$$
\pi_C > \pi_A \wedge \pi_C \leq x + \pi_A \wedge \pi_C > \frac{1+x}{2} - \frac{\pi_A}{2} \wedge \pi_B \geq 0
$$

Note that $a + b + c = O$ and $a, b, c \geq 0$, imply that $\pi_A + \pi_C \leq 1$.

ii) Similarly, from $c > b \geq c - M > a \geq 0$, we obtain that the result must hold whenever

$$\pi_C > \frac{1}{2} - \frac{\pi_A}{2} \wedge \pi_C \leq \frac{1+x}{2} - \frac{\pi_A}{2} \wedge \pi_C > x + \pi_A \wedge \pi_A \geq 0$$

iii) Finally, from $(a \geq b \wedge M + b > a \wedge M + b \geq c \wedge c \geq 0) \vee (a < b \wedge M + a \geq c \wedge M + a \geq b \wedge c \geq 0)$, we get that the result must hold whenever

$$\begin{aligned}(\pi_C &\geq 1 - 2\pi_A \wedge \pi_C < \frac{1+x}{2} - \frac{\pi_A}{2} \wedge \pi_C \leq 1 + x - 2\pi_A \\ \vee \pi_C &< 1 - 2\pi_A \wedge \pi_C \leq x + \pi_A \wedge \pi_C \geq 1 - x - 2\pi_A) \wedge \pi_C \geq 0\end{aligned}$$

■

Note that the above Proposition does not characterize the limits of the above probabilities in all the euclidean zero-measure frontiers of the sets E_{jk}, $Front(E_{jk})$. However, the following Proposition allows for a characterization of RBR almost everywhere in the simplex.

Proposition 6 completes Proposition 5, in the sense that it states how an AB type rhizomatically best responds, as a function of π_A and π_C, in the regions where all three probabilities, p^N_{AC}, p^N_{CB} and p^N_{AB} go to zero as $N \to \infty$.

Again, with no loss of generality, we state it for an AB type with $q > 0$. Let:

$$\begin{aligned}E_1 &= \{(\pi_A, \pi_C) \in \Delta \backslash \{(0,0)\} : \pi_C \leq \pi_B \wedge \pi_B \geq x + \pi_A \wedge x < 1\} \\ E_2 &= \{(\pi_A, \pi_C) \in \Delta \backslash \{(0,1)\} : \pi_C \geq x + \pi_B \wedge \pi_C \geq x + \pi_A \wedge x < 1\} \\ E_3 &= \{(\pi_A, \pi_C) \in \Delta : \pi_C \leq \pi_A \wedge \pi_A \geq x + \pi_B \wedge x < 1\}\end{aligned}$$

Proposition 6 *Let Assumptions $1-2$ hold. Let voter i be an AB-type with $q > 0$. Asymptotic RBR (as $N \to \infty$) are as follows:*

i) If $ql_{AB} \geq \frac{1}{2} \Leftrightarrow x = \frac{ql_{AB}}{1-ql_{AB}} \geq 1$, an AB-type will vote for A.

ii) Let $(\pi_A, \pi_C) \in E_1$. If $ql_{AB} \geq \frac{1}{3}$, an AB-type will vote for A. If $ql_{AB} < \frac{1}{3} \Leftrightarrow x = \frac{ql_{AB}}{1-ql_{AB}} < \frac{1}{2}$, let $\overline{q}_{AB}$ be given implicitly and uniquely by the equation

$$\begin{aligned}&\ln \frac{\pi_C + \sqrt{x^2\pi_C^2 - 4\pi_A\pi_B x^2 + 4\pi_A\pi_B}}{(1-x^2)\left(\pi_A + 2\sqrt{\pi_B\pi_C}\right)} \\ = \; &x \ln \frac{1}{2\pi_B - 2x\pi_B}\left(x\pi_C + \sqrt{x^2\pi_C^2 - 4\pi_A\pi_B x^2 + 4\pi_A\pi_B}\right)\end{aligned} \tag{3}$$

then, if $\pi_A > 0$, an AB type will vote for B whenever $q < \overline{q}_{AB}$ and will vote for A whenever $q > \overline{q}_{AB}$. If $\pi_A = 0$, an AB type will vote for B.

iii) Let $(\pi_A, \pi_C) \in E_2$. If $ql_{AB} \geq \frac{1}{2}$, an AB-type will always vote for A. If $ql_{AB} < \frac{1}{2}$, an AB type will vote for B whenever $\pi_B > \pi_A$, and will vote for A whenever $\pi_B < \pi_A$.

iv) Let $(\pi_A, \pi_C) \in E_3$. Then, an AB-type will always vote for A.

v) If $(\pi_A, \pi_C) = (0,0)$ or $(\pi_A, \pi_C) = (0,1)$, an AB-type votes A.

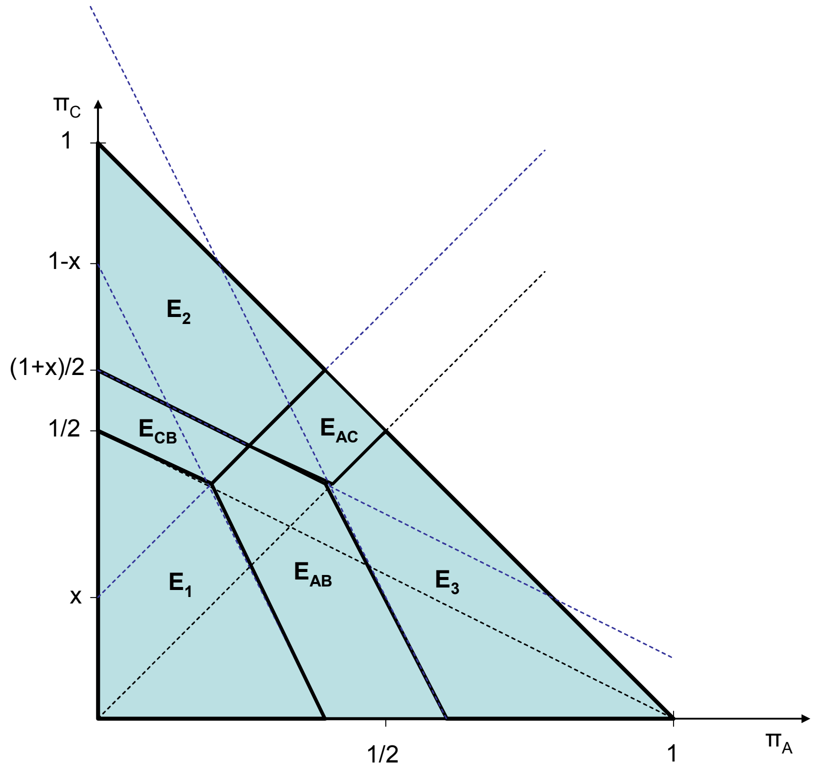

Figure 1: The diagram represents the partition of the simplex into the relevant regions for Propositions 5 and 6.

Notice that when $x = \overline{x} = 0$, 3 simplifies to $\frac{\pi_C + 2\sqrt{\pi_A \pi_B}}{\pi_A + 2\sqrt{\pi_B \pi_C}} = 1 \Leftrightarrow \pi_C + 2\sqrt{\pi_A \pi_B} = \pi_A + 2\sqrt{\pi_B \pi_C} \Leftrightarrow \pi_C = \pi_A$, $\forall (\pi_A, \pi_C) \in E_1$, and so Proposition 6 holds also for $q = 0$, in which case it captures the results from Palfrey [21].

The proof of Proposition 6 proceeds in several steps and is presented in appendix B. However, since it relies on very different techniques from those used in Palfrey [21], it is instructive to sketch the main steps of the proof in the text.

Sketch of the proof of Proposition 6: Let $(\pi_A, \pi_C) \in E_1 \cap int\Delta$ and $ql_{AB} < \frac{1}{3}$. From 2, an AB type will vote for A as long as $\lambda_{AB} \leq (1 + \frac{p^N_{AC}}{p^N_{AB}})/(1 + \frac{p^N_{CB}}{p^N_{AB}})$. In order to characterize asymptotic RBR, we need to compute $\lim_{N \to \infty} (1 + \frac{p^N_{AC}}{p^N_{AB}})/(1 + \frac{p^N_{CB}}{p^N_{AB}})$, in a set in which all three probabilities go to zero as N grows large. Furthermore, we cannot use the approximation of the trinomial distribution to the bivariate normal distribution, since this approximation is valid only when the deviation of j_N from $O\pi_j$ is of the order of $\sqrt{O}$,

$j = a, b, c$, and in our setting this deviation is of order O. We are thus in the domain of large deviations/rare events, in which large deviations theory provides us the adequate tools for obtaining asymptotically accurate results. Appendix A contains the definitions and results from large deviations theory that are used in this essay.

Let $j' \geq 0$ be the sample mean of j_N, $j' = \frac{j_N}{O}$, $j_N = a_N, b_N, c_N$. Then $a' + b' + c' = 1$. Let

$$I(a', c') = a' \ln \frac{a'}{\pi_A} + (1 - a' - c') \ln \frac{(1 - a' - c')}{\pi_B} + c' \ln \frac{c'}{\pi_C} \tag{4}$$

be the (unique) rate function of the trinomial distribution ($b' = 1 - a' - c'$ was used in the above expression, and $0 \ln 0$ is conventionally defined to be 0).

Working with the rate function of the trinomial distribution and applying Varadhan's Theorem (usually referred to in the literature as Varadhan's Lemma), we establish the following Lemma:

Lemma 7 *Let Assumptions* $1 - 2$ *hold. Let voter* i *be an* AB *type with* $q > 0$*. Let* $\pi_j > 0, j = A, B, C$*. Let* $(\pi_A, \pi_C) \notin E_{jk}$*. Then,*

$$\lim_{O \to \infty} \frac{1}{O} \ln p_{jk}^N = \sup_{(a', c') \in E_{jk}} - I(a', c') \text{ [7]} \tag{5}$$

Using Stirling's approximation,

$$O! \simeq \sqrt{2\pi O} \left(\frac{O}{e} \right)^O$$

it is straightforward to establish that the probability function

$$f(a', c') = f(a_N, c_N) = \frac{O!}{a_N!(O - a_N - c_N)!c_N!} \pi_A^{a_N} \pi_B^{O - a_N - c_N} \pi_C^{c_N}$$

verifies the following identity

$$\lim_{O \to \infty} \frac{1}{O} \ln f(.,.) = -I(.,.). \tag{6}$$

Since I is strictly convex, 5 provides a simple way of computing the rate at which all three probabilities tend to zero, by solving optimization problems with unique optimizers. Varadhan's Lemma gives expression to the intuitive fact that, asymptotically, almost all of the mass of p_{jk}^N lies in the "closest" point in E_{jk}, as measured by the statistical distance $I(.,.)$, to $(\pi_A, \pi_C) \in E_1 \cap int\Delta$. Thus, unless $\sup_{(a', c') \in E_{CB}} - I(a', c') = \sup_{(a', c') \in E_{AB}} - I(a', c')$, which happens when 3 holds, p_{CB}^N and p_{AB}^N will converge to zero at different rates. More precisely, if $\overline{q}$ is given by 3, we have

[7] There is an abuse of notation here: E_{jk} was originally defined as a set of equilibrium probabilities, (π_A, π_C); here, the variables are in fact (a', c').

$$\underset{N\to\infty}{lim}\frac{p_{CB}^N}{p_{AB}^N}=\left\{\begin{array}{c}\infty, q<\overline{q}\\ 0, q>\overline{q}\end{array}\right. .$$

Noting that $\underset{N\to\infty}{lim}\frac{p_{AC}^N}{p_{AB}^N}=0$, since any point in E_{AC} is more "distant" that any point in E_{AB}, for all $(\pi_A,\pi_C)\in E_1\cap int\Delta$, we obtain the cutoff $\overline{\lambda}_{AB}$ such that

$$\overline{\lambda}_{AB}=\lim_{N\to\infty}\frac{1+\frac{p_{AC}^N}{p_{AB}^N}}{1+\frac{p_{CB}^N}{p_{AB}^N}}=\left\{\begin{array}{c}0, q<\overline{q}\\ 1, q>\overline{q}\end{array}\right.$$

Hence, an AB type will vote for A if $q>\overline{q}$ and for B if $q<\overline{q}$.

For $ql_{AB}\geq\frac{1}{3}$, it is easy to establish that an AB type always votes for A (see appendix B). Finally, the validity of the result is straightforwardly established for the cases where $\pi_A=0$ or $\pi_C=0$.

For $(\pi_A,\pi_C)\in E_2$ the steps are identical, only now the solution of $\sup_{(a',c')\in E_{CB}}-I(a',c')=\sup_{(a',c')\in E_{AC}}-I(a',c')$ is simply $\pi_B=\pi_A$. Also, the cutoff $\overline{\lambda}_{AB}$ is:

$$\overline{\lambda}_{AB}=\lim_{N\to\infty}\frac{\frac{p_{AB}^N}{p_{CB}^N}+\frac{p_{AC}^N}{p_{CB}^N}}{\frac{p_{AB}^N}{p_{CB}^N}+1}=\left\{\begin{array}{c}\infty, \pi_A>\pi_B\\ 0, \pi_A<\pi_B\end{array}\right.$$

Finally, if $(\pi_A,\pi_C)\in E_3$, it is immediate that if an AB type votes for A, A always wins. This ends the sketch of the proof.

Gathering Propositions 5 and 6 immediately establishes our main result for RBR.

Let $E=\{(\pi_A,\pi_C)\in\Delta:\pi_C<1-2\pi_A\wedge\pi_C>x+\pi_A\}\cup\{(\pi_A,\pi_C)\in E_1:q<\overline{q}_{AB}\}$, where $\overline{q}_{AB}$ is as defined in Proposition 6.

Theorem 8 *Let Assumptions $1-2$ hold. Let voter i be an AB type with $q>0$. Asymptotic RBR (as $N\to\infty$) are as follows:*

i) Let $(\pi_A,\pi_C)\in E$. Then, an AB type will vote for B a. e.[8].
ii) Let $(\pi_A,\pi_C)\in\Delta/extE$. Then, an AB type will vote for A.

Figure 2 illustrates RBR as characterized by the Theorem.

The Theorem above ensures that, in particular, an AB type never votes for B whenever $\pi_A>\min\{\pi_B,\pi_C\}$, which is a feature that our model shares with Palfrey's, as it intuitively should. Let $\pi_A<\min\{\pi_B,\pi_C\}$, and in specific, $\pi_A<\pi_B\leq\pi_C$, with no loss of generality[9]. The cutoffs for $\overline{\lambda}_{Aj}$, $j\in\{B,C\}$,

[8] RBRs are completely characterized in proposition 2, with the exception of the curves that give the cutoffs $\overline{q}_{AB}$.

[9] The case $\pi_A=\min\{\pi_B,\pi_C\}$ is analyzed in the next subsection.

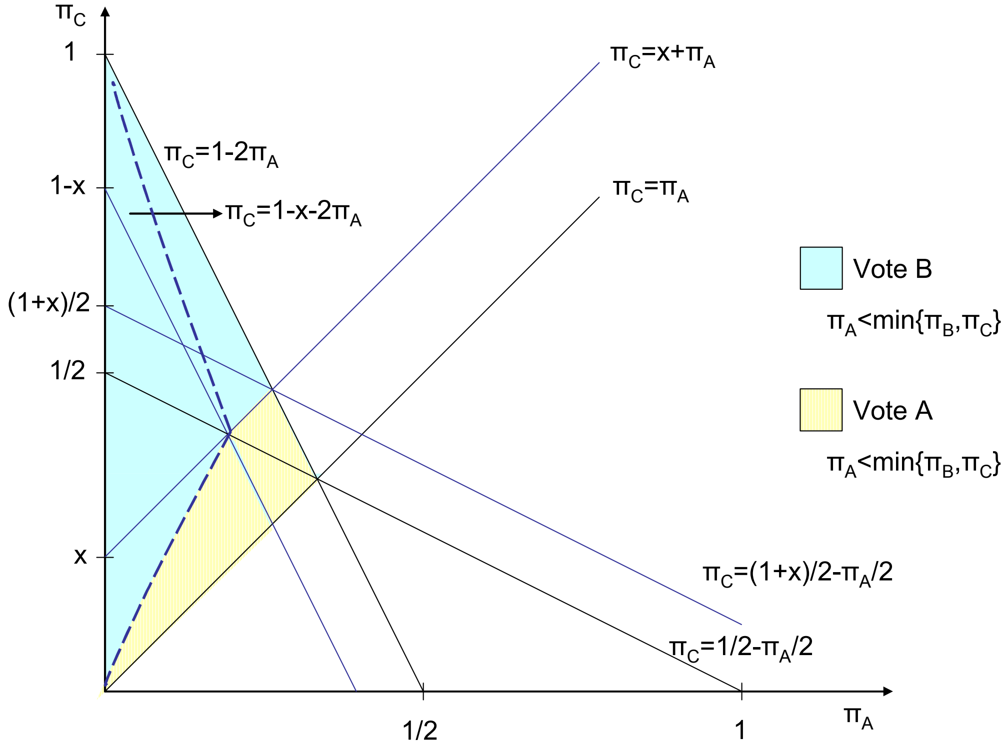

Figure 2: Best responses for an $AB-$type

are always either 0 or greater than or equal to 1, but $\overline{\lambda}_{Aj}$ is a function of q_{Aj}, parametrized by $\overline{q}_{Aj}$[10]: the equations that confirm equilibrium probabilities therefore depend only on $G_{Aj}(\overline{q}_{Aj})$ and are given in our second main technical result.

Let $\overline{q}_{AC} = \frac{\overline{x}_{AC}}{l_{AC}(1+\overline{x}_{AC})}$, with $\overline{x}_{AC} \in [0,1)$, be (uniquely) determined by

$$\begin{aligned} & \ln \frac{\pi_B + \sqrt{\overline{x}_{AC}^2 \pi_B^2 - 4\pi_A \pi_C \overline{x}_{AC}^2 + 4\pi_A \pi_C}}{\left(1 - \overline{x}_{AC}^2\right)\left(\pi_A + 2\sqrt{\pi_B \pi_C}\right)} \qquad (7) \\ = \; & \overline{x}_{AC} \ln \frac{1}{2\pi_C - 2\overline{x}_{AC}\pi_C} \left(\overline{x}_{AC}\pi_B + \sqrt{\overline{x}_{AC}^2 \pi_B^2 - 4\pi_A \pi_C \overline{x}_{AC}^2 + 4\pi_A \pi_C}\right) \end{aligned}$$

Theorem 9 *Let Assumptions $1-2$ hold. Let (π_A, π_B, π_C) be the equilibrium probabilities of a RNE, such that $\pi_A < \pi_B \leq \pi_C$. Let $\overline{q}_{AC}$ be given uniquely by 7 if $\pi_A > 0$ and $\overline{q}_{AC} = \frac{1}{3l_{AC}}$ if $\pi_A = 0$. Then,*

[10] Note that all the results hold even if cutoffs are greater than 1.

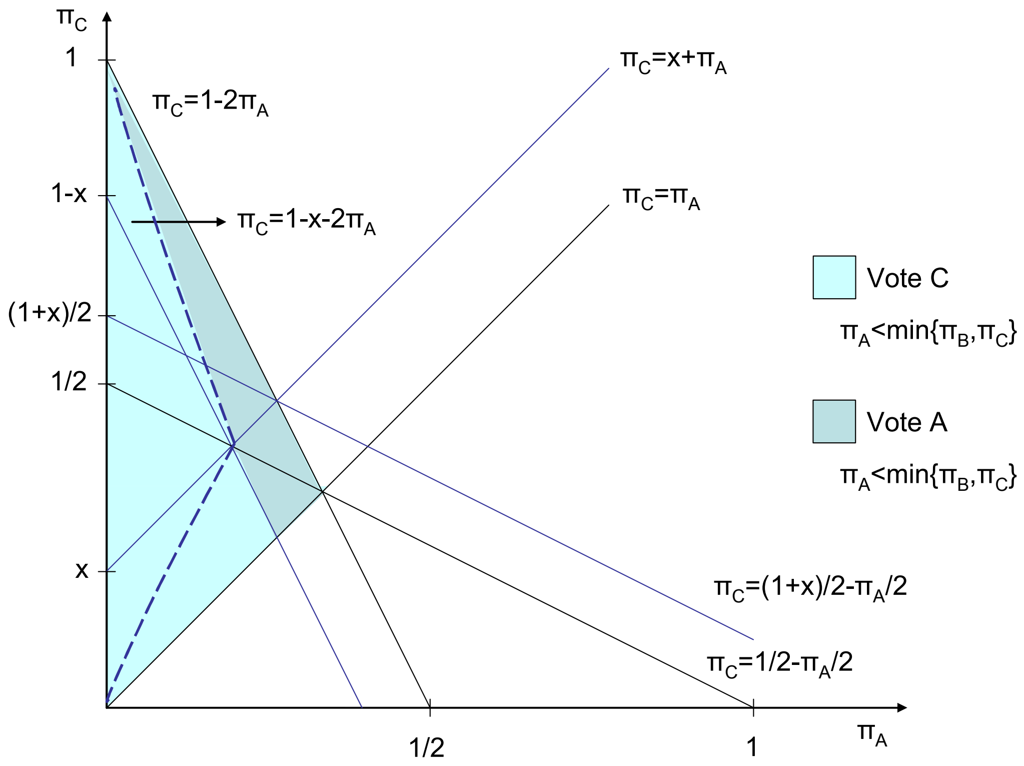

Figure 3: Best responses for an AC-type

$$
\begin{aligned}
\pi_A &= l_{AB}(1 - G_{AB}(\frac{\pi_C - \pi_A}{l_{AB}(1 + \pi_C - \pi_A)})) + l_{AC}(1 - G_{AC}(\overline{q}_{AC})) \qquad (8)\\
\pi_B &= l_{BA} + l_{BC} + l_{AB}G_{AB}(\frac{\pi_C - \pi_A}{l_{AB}(1 + \pi_C - \pi_A)})\\
\pi_C &= l_{CA} + l_{CB} + l_{AC}G_{AC}(\overline{q}_{AC})
\end{aligned}
$$

Proof. Let $E' = \{(\pi_A, \pi_C) \in \Delta : \pi_A < \pi_B \leq \pi_C\}$. From Propositions 5 and 7, an AB type votes B if $(\pi_A, \pi_C) \in E' \cap (E_{CB} \cup E_2)$ and $q_{AB}l_{AB} < \frac{1}{2}$ and votes A if $(\pi_A, \pi_C) \in E_{AB}$ or $q_{AB}l_{AB} \geq \frac{1}{2}$. Thus, an AB type votes A if $x_{AB} > \pi_C - \pi_A \Leftrightarrow q_{AB} > \frac{\pi_C - \pi_A}{l_{AB}(1+\pi_C-\pi_A)}$ and votes B if $q_{AB} < \frac{\pi_C - \pi_A}{l_{AB}(1+\pi_C-\pi_A)}$. The results for $\overline{q}_{AC}$ obtain directly from Proposition 6, given the symmetry of the results for q_{AB} and q_{AC} in the exchange of π_B for π_C and vice-versa. ■

On one hand, we can check from this characterization of the equilibria that strategic voting for the second best candidate will be (weakly) decreasing with respect to rhizomatic beliefs. In rough terms, a more rhizomatic agent will tend to vote more often for her preferred candidate.

On the other hand, this Theorem will allow us to calculate the set of possible equilibria for each polity. In fact, an equilibrium always exists.

Lemma 10 *Under Assumptions* 1 *and* 2, *there is at least one Duvergerian RNE for each polity.*

Proof. It is straightforward to check that, under Assumption 1, there are at least four types jk such that $j,k \in \{A,B,C\} : j \neq k$ for which $l_{jk} < \frac{1}{3}$[11]. In particular, there must be at least one j such that $l_{jk} < \frac{1}{3}$ and $l_{jl} < \frac{1}{3}$. Without loss of generality, let $l_{AB} < \frac{1}{3}$ and $l_{AC} < \frac{1}{3}$.

There are three possible Duvergerian outcomes. Again without loss of generality, let $l_{BA} + l_{BC} + l_{AB} < l_{CA} + l_{CB} + l_{AC}$. The Duvergerian outcome where A loses would then be such that $\pi_A < \pi_B \leq \pi_C$, $\pi_A = 0$ and $\pi_C > \frac{1}{2}$. We have that $\overline{q}_{AC} = \frac{1}{3l_{AC}} > 1$ and therefore $G_{AC}(\overline{q}_{AC}) = 1$. Moreover, $\frac{\pi_C - \pi_A}{l_{AB}(1+\pi_C-\pi_A)} = \frac{\pi_C}{l_{AB}(1+\pi_C)} > 1 \Leftrightarrow \pi_C > \frac{l_{AB}}{1-l_{AB}}$ since $\frac{l_{AB}}{1-l_{AB}} < \frac{1}{2} < \pi_C$. Therefore, $G_{AB}(\frac{\pi_C-\pi_A}{l_{AB}(1+\pi_C-\pi_A)}) = 1$. But then from 8 we have $\pi_A = 0$, $\pi_B = l_{BA} + l_{BC} + l_{AB}$ and $\pi_C = l_{CA} + l_{CB} + l_{AC}$ and the equilibrium holds. ∎

We therefore conclude that at least one Duvergerian equilibrium will always remain in this setting. On one hand, this seems to reinforce Palfrey's [21] conclusion. On the other hand, Palfrey concluded that there would always be three two-party equilibria such that each party would lose in exactly one of them regardless of the preferences. Our next step is to investigate whether equilibrium selection is possible in our model and whether non-Duvergerian outcomes can also arise.

Notice that a possible computational drawback of our model might result from the fact that 7 does not have a closed form solution for threshold $\overline{q}_{AC}$. However, numerical computations need not allow for general distributions of RB. Since RNE conditions depend only on $G_{jk}(\overline{q}_{jk})$, $j = A,B,C$, the properties of RNE can be analyzed, with no loss of generality, using only simple discrete distributions $H_{jk}(.)$ for which

$$H_{jk}(q_{jk}) = \begin{cases} p_{jk} & q_{jk} = 0 \\ 1 - p_{jk} & q_{jk} = 1 \end{cases} \tag{9}$$

This procedure brings about two main advantages: the simplification of RNE computations, since 8 allow for a straightforward check of RNE conditions; and most importantly, an analysis of implications of possible correlations among types and beliefs.

After equilibrium probabilities are computed, together with values for the relevant cutoffs, these will remain the equilibria for any distribution that leads to the same accumulated probability mass at each cutoff.

[11] If Assumption 1 were relaxed, we could have polities for which only three types would satisfy $l_{jk} < \frac{1}{3}$. However, the result would still hold a.e.

4 Examples and applications

This section analyzes RNE properties for some pertinent examples and draws general insights from our model.

Example 1

Let $l_{CA} \geq \frac{1}{2}$ and $p_{CA} = 0$. From Theorem 1, it is clear that a CA type votes $C \Rightarrow \pi_C \geq \frac{1}{2}$. C therefore wins, regardless of what A and B types do.

Therefore, if there is a majority of agents with the same preference ordering, the Condorcet winner will be selected as long as the agents are rhizomatic enough. Whereas Palfrey [21] concludes that any candidate can lose (with zero votes) in a large election under Plurality Rule, our model eliminates the possibility of candidate C being defeated.

Example 2

Let $p_{jk} = 0$ for all $j, k \in \{A, B, C\} : j \neq k$. Consider the following polity:

l_{AB}	l_{AC}	l_{BA}	l_{BC}	l_{CA}	l_{CB}
0.01	0.14	0.4	0	0.45	0

In this case, preferences are single-peaked and we have a unique equilibrium characterized by:

$$\begin{cases} \pi_A = 0 \\ \pi_B = 0.41 \\ \pi_C = 0.59 \end{cases} \text{and } \overline{q}_{AC} \simeq 2.381 \text{ and } \frac{\pi_C - \pi_A}{l_{AB}(1 + \pi_C - \pi_A)} \simeq 37.107.$$

On one hand, this example suffices to show that equilibrium selection can arise depending on the preferences and rhizomatic beliefs. On the other hand, this example shows us that a Condorcet winner may not win in the unique equilibrium prediction: in fact, A would be the Condorcet winner in this case.

Example 3

Let $p_{jk} = 0$ for all $j, k \in \{A, B, C\} : j \neq k$. Consider the following polity:

l_{AB}	l_{AC}	l_{BA}	l_{BC}	l_{CA}	l_{CB}
0.15	0.18	0.3	0.01	0.35	0.01

In this case, we have three equilibria, one of which is not Duvergerian:

$$\begin{cases} \pi_A = 0.33 \\ \pi_B = 0.3 \\ \pi_C = 0.37 \end{cases} \text{and } \overline{q}_{BC} \simeq 1.501 \text{ and } \frac{\pi_C - \pi_B}{l_{BA}(1 + \pi_C - \pi_B)} \simeq 0.218.$$

We can conclude that even though at least one equilibrium satisfies Duverger's Law, there may be equilibria where Duverger's Law may fail to hold. In this case, BA types vote B because they are rhizomatic; otherwise, they would vote A and lead to the victory of the Condorcet winner.

Example 4

Let $p_{jk} = 0$ for all $j, k \in \{A, B, C\} : j \neq k$. Consider the following polity:

l_{AB}	l_{AC}	l_{BA}	l_{BC}	l_{CA}	l_{CB}
0.04	0.18	0.1	0.21	0.34	0.13

In this case, we have two possible equilibria, one of which is not Duvergerian:

$$\begin{cases} \pi_A = 0.18 \\ \pi_B = 0.35 \\ \pi_C = 0.47 \end{cases} \text{and } \overline{q}_{AC} \simeq 0.780 \text{ and } \frac{\pi_C - \pi_A}{l_{AB}(1+\pi_C-\pi_A)} \simeq 5.620 \text{ and}$$

$$\begin{cases} \pi_A = 0 \\ \pi_B = 0.35 \\ \pi_C = 0.65 \end{cases} \text{and } \overline{q}_{AC} \simeq 1.852 \text{ and } \frac{\pi_C - \pi_A}{l_{AB}(1+\pi_C-\pi_A)} \simeq 9.848$$

In this case, we can confirm that the model allows for equilibrium selection and for non-Duverger outcomes. Moreover, even though uniqueness of equilibrium is lost, we do have a unique prediction with respect to the ordering of candidates.

Example 5

Let $p_{CA} = p_{CB} = 0$ and let $p_{jk} = 1$ for $j \in \{A, B\}$, $k \in \{A, B, C\} : k \neq j$.

Consider the following polity:

l_{AB}	l_{AC}	l_{BA}	l_{BC}	l_{CA}	l_{CB}
0.1	0.28	0.06	0.2	0.01	0.35

In this case, we have two Duvergerian equilibria:

$$\begin{cases} \pi_A = 0 \\ \pi_B = 0.36 \\ \pi_C = 0.64 \end{cases} \text{and } \overline{q}_{AC} \simeq 1.190 \text{ and } \frac{\pi_C - \pi_A}{l_{AB}(1+\pi_C-\pi_A)} \simeq 3.902 \text{ and}$$

$$\begin{cases} \pi_A = 0.44 \\ \pi_B = 0 \\ \pi_C = 0.56 \end{cases} \text{and } \overline{q}_{BC} \simeq 1.667 \text{ and } \frac{\pi_C - \pi_B}{l_{BA}(1+\pi_C-\pi_B)} \simeq 5.983.$$

This example is particularly informative with respect to the importance of the correlations between preferences and beliefs. If CA-types and $CB-$types were not rhizomatic, any Duvergerian equilibrium might arise. The equilibrium where C would be defeated is ruled out due to the rhizomatic beliefs. Whereas agents who prefer candidates A or B are easily willing to vote for their second preference, that is not the case with voters who prefer C. Therefore, it is possible for C to ensure the win by emphasizing group identity and stimulating rhizomatic thinking. In many instances, this is the type of behavior that can be observed in campaigns that call for strategic voting.

It is worth remarking that a reversal of l_{AB} and l_{AC} would lead the first equilibrium to disappear and be replaced by an equilibrium where:

$$\begin{cases} \pi_A = 0 \\ \pi_B = 0.54 \\ \pi_C = 0.46 \end{cases} \quad \text{and } \overline{q}_{AB} \simeq 1.190 \text{ and } \frac{\pi_B - \pi_A}{l_{AC}(1+\pi_B-\pi_A)} \simeq 3.506.$$

The main conclusion we can derive from this example is that the distribution of preferences is also crucial to realize whether political strategies should strive to foster group identity around a given candidate.

Example 6: The 1992 US Presidential Election

In the 1992 US presidential election, Bill Clinton (C) received 43% of the popular vote, George Bush (B) 37,4% and Ross Perot (P) 18,9%, in a total of 99.3%.

The US president is elected with at least 270 votes from the Electoral College and not on a simple PR basis, which may have repercussions on strategic voting behavior. Still, in 1992 the Electoral College elected representatives in each state on a winner-takes-all simple PR basis. Furthermore, inspection of the popular vote in each state reveals that Duverger's Law failed to hold in the large majority of states. In Iowa, for instance, the results were almost an exact match of the nationwide popular vote: 43.3% of the votes for Clinton, 37.3% for Bush and 18.7% for Perot in a total of 99, 3%.

For simplicity, and since qualitative results do not depend on the exact equilibrium probabilities, we use nationwide results in what follows. The purpose of this exercise is to analyze how RB might help explain the observed equilibrium and identify the determinants for such voting behavior.

Note that, in empirical cases, equilibrium probabilities are known instead of the distribution of preferences: we discuss the assumptions we make on preferences after we analyze the example. Furthermore, and since there were other candidates receiving votes, an exact analysis would require the use of a multinomial model with more than three alternatives: for this discussion, we normalize equilibrium probabilities to add up to 1 in the trinomial model. Let then $\pi_C = 0.433$, $\pi_B = 0.377$, and $\pi_P = 0,190$ denote the equilibrium probabilities.

It has been argued that Perot may have been partly responsible for Bush's defeat. The claim is that a majority of PB types still voted for Perot, whereas the majority of PC types shifted their vote to Clinton. Applying the previous results to these equilibrium probabilities, Theorem 9 yields $\frac{\pi_C - \pi_P}{l_{PB}(1+\pi_C-\pi_P)} = \frac{0.433-0.19}{l_{PB}(1+0.433-0.19)}$ and from 7 $\overline{x}_{PC} = 0.19602 \Leftrightarrow \overline{q}_{PC} l_{PC} = 0.16389$. Hypothesize, for instance, that $l_{BP} + l_{BC} = 0.36$, $l_{CP} + l_{CB} = 0.34$, $l_{PB} = 0.207$ and $l_{PC} = 0.093$. Then, all PC types have $q_{PC} < \frac{0.163\,89}{l_{PC}} = \frac{0.163\,89}{0.093} = 1.7623$, so that $\pi_C = l_C + l_{PC} = 0.433$. Also, $\pi_B = l_B + \alpha l_{PB} = 0.36 + 0.207\alpha = 0.377 \Leftrightarrow \alpha = 0.082$. Since $\frac{\pi_C - \pi_P}{l_{PB}(1+\pi_C-\pi_P)} = 0.94444$, as long as $G_{PB}(0.944\,44) = 0.082$, the equilibrium is sustained.

Note that, clearly, a Duvergerian outcome would give the victory to Bush, since $l_{BP} + l_{BC} + l_{PB} = 0.36 + 0.207 = 0.567 > l_{CP} + l_{CB} + l_{PC} = 0.433$. Also, under sincere voting, Clinton would also have lost to Bush: $l_C = 0.34 < l_B = 0.36$.

Another interesting feature of the hypothesized distribution of preferences, is that the fraction of Perot supporters who prefer Clinton in second place are less than half of those that prefer Bush in second place. The key is that they are just enough voters to make Clinton win, but also few enough voters such that they vote Clinton no matter how rhizomatic they may be. In fact, if this number were large enough, rhizomatic PC types might prefer to vote Perot, breaking the equilibrium. It is worth stressing that what allows Clinton to win in this scenario is not just the fact that there are just enough PB types but also that most of them are very rhizomatic.

Remembering that RT captures the notion of group-identity, and assuming the hypothesized distribution of preferences to hold approximately, the outcome of the 1992 US presidential election might be at least partially explained by this rationale: Most of Clinton and Bush supporters ranked Perot second and had a strong feeling of group-identity. A majority of Perot supporters preferred Bush to Clinton, again with a strong feeling of group-identity. The number of Perot supporters preferring Clinton to Bush was smaller but nonetheless high enough to make them anticipate they were more likely to be pivotal for Clinton than for Perot. In that case, PC types should shift their vote to Clinton and PB types stick with Perot. And this may indeed be part of the story of the 1992 US presidential elections.

We can conclude from these examples that:

i) for given distributions of types, there is equilibrium selection, in the sense that some of Palfrey's two-party equilibria are eliminated; the equilibrium may be unique (and there are also cases where the equilibrium winner can be uniquely determined even if uniqueness of equilibrium is lost); namely, whenever at least half of the population share the same preferences, then their preferred candidate wins the election as long as those voters are sufficiently rhizomatic;

ii) Duverger's Law may break and that can help explain empirically observed equilibrium outcomes;

iii) Correlations between preferences and beliefs are the key determinants of equilibria. In any case, the diversity of possible equilibrium outcomes that may result from combining different distributions of preferences and RB requires further analysis.

5 Conclusions

We introduce rhizomatic thinking in a pivotal-voter model under Plurality Rule. This is to the best of our knowledge the first time an alternative to the assumption of individual rationality is introduced in pivotal-agent games. Our rhizomatic assumption allows any agent i to have an exogenous rhizomatic belief q_i as to the proportion of types with the same preference ranking that i believes will take the same action as she does[12]. Among several other behavioral assumptions, RT captures the feeling of group-identity, and provides a horizontal notion of rationality capable of encompassing individual as well as more collective notions of rationality. This assumption impacts strategic behavior, since now an agent may perceive her pivot probabilities to be non-vanishing as the number of voters grows large. We find that *caeteris paribus* vote manipulation is (weakly) decreasing in rhizomatic beliefs.

In our model, equilibria always exist and can be determined for any distribution of preferences and rhizomatic beliefs. In particular, unique equilibrium outcomes result for certain distributions of preferences and types. Namely, if an alternative is preferred by at least one half of the population with the same preference ordering, then it will always be selected if that segment of the population is fully rhizomatic. This conclusion eliminates Duvergerian equilibria in which such an alternative is defeated.

A key result of our model is that Duverger's Law is violated: there are polities with non-Duvergerian equilibrium outcomes. This result cannot be derived from existing models in which individual rationality is not relaxed but it allows us, for instance, to provide a rationale for the three-party equilibrium outcome of the 1992 US presidential election.

Our model also suggests that there is room for an in-depth analysis of correlations between preferences and rhizomatic beliefs, and that this analysis may prove to be useful for political strategists who may or may not wish to enhance feelings of group identity and to call for strategic voting

Other directions for future research include a global games approach that would allow for uncertainty with respect to each agent's own rhizomatic belief and own class of like-minded others, leading RBR to have a statistical nature rather than a deterministic one.

Also, the methodology we present for the determination of asymptotic RNE can be generalized to any number of candidates, provided the multinomial distribution is used.

More importantly, the results presented for PR can be both positively and normatively compared with those for other aggregation rules such as Approval Voting (AV). Palfrey argues that the implications of strategic voting are not well understood for AV. Although this claim was made before the contributions of Roger Myerson towards a better understanding of the implications of strategic voting under AV in Poisson-Myerson environments, it is still true that there is no direct comparison in the literature between PR and AV in a multinomial setting. In a companion paper, we provide such a comparison for both individually rational and rhizomatic agents.

[12]We assumed the class of like-minded others to be the class of agents with the same preference ranking.

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A On Large Deviations

We first state Varadhan's Lemma.

The reader can find the omitted definitions on den Hollander [14], for instance.

Varadhan's Lemma: Let (P_N) be a sequence of probability measures that satisfy the Large Deviation Principle (LDP) on χ with rate N and rate function I. Let $F : \chi \to \mathbb{R}$ be a continuous function that is bounded from above. Then,

$$\lim_{N\to\infty} \frac{1}{N} \ln \int_{\chi} \exp(NF(x))P_N(dx) = \sup_{x\in\chi} [F(x) - I(x)]$$

We now establish the main result from large deviations to be used in the proof of Proposition 6.

Lemma 11 *Let Assumptions* $1-2$ *hold. Let voter* i *be an* AB *type with* $q > 0$. *Let* $\pi_j > 0, j = A, B, C$. *Let* $(\pi_A, \pi_C) \notin E_{jk}$. *Then,* $\lim\limits_{O\to\infty} \frac{1}{O} \ln p_{jk}^N = -\sup\limits_{(a',c')\in E_{jk}} I(a', c')$.

Proof. Let $\chi = E_{jk}$. Let $F(x) = 0$. Let (P_N) be the sequence of probability measures that assigns probability $f_N(a_N, c_N) = \frac{O!}{a_N!(O-a_N-c_N)!c_N!} \pi_A^{a_N} \pi_B^{O-a_N-c_N} \pi_C^{c_N}$ to each corresponding $(\frac{a_N}{O}, \frac{c_N}{O}) \in E_{jk}$, and zero otherwise. Let $a' = \frac{a_N}{O}, c' = \frac{c_N}{O}$. Then, $I(a', c') = a' \ln \frac{a'}{\pi_A} + (1 - a' - c') \ln \frac{(1-a'-c')}{\pi_B} + c' \ln \frac{c'}{\pi_C}, (a', c') \in E_{jk}$, is known to be the unique (strictly convex, lower semi-continuous with compact level sets) rate function associated with (P_N)[13].

Let $I(S) = \inf\limits_{x\in S} I(x), \forall S \subset \Delta$.

[13] See, for instance, den Hollander [14], pp. 29-34.

Since $\underset{O\to\infty}{lim}\frac{1}{O}\ln f(.,.) = -I(.,.) \Rightarrow \underset{O\to\infty}{\limsup}\frac{1}{O}\ln P_N(C) \leq -I(C), \forall C \subset \Delta$ closed $\wedge \underset{O\to\infty}{\liminf}\frac{1}{O}\ln P_N(D) \geq -I(D), \forall D \subset \Delta$ open, (P_N) satisfies the LDP on Δ[14].

Then, we can apply Varadhan's' Lemma to obtain
$\lim_{O\to\infty}\frac{1}{O}\ln p_{jk}^N = \lim_{O\to\infty}\frac{1}{O}\ln\int_{E_{jk}} P_N(dx) = \sup_{(a',c')\in E_{jk}} -I(a',c')$. ■

B Proof of Proposition 6

Proof. Let $I(a',c') = a'\ln\frac{a'}{\pi_A} + (1-a'-c')\ln\frac{(1-a'-c')}{\pi_B} + c'\ln\frac{c'}{\pi_C}$ be the (unique) rate function of the trinomial distribution, where $j' \geq 0$ is the sample mean of j_N, $j' = \frac{j_N}{O}$, $j_N = a_N, b_N, c_N$, $a' + b' + c' = 1$, where $b' = 1 - a' - c'$ was used in the expression for I, and $0\ln\frac{0}{\pi_j}$ is conventionally defined to be 0 for all $\pi_j \in [0,1]$.

i) If $M_{AB}^N \geq O_{AB}^N$, voting for A yields $a + M_{AB}^N \geq \max\{b,c\}$, where equality obtains only if $a = 0$, $\max\{b,c\} = O_{AB}^N$ and $M_{AB}^N \geq O_{AB}^N$, and voting for B would be a dominated action. Since $\lim_{N\to\infty}\frac{M_{AB}^N}{O_{AB}^N} = \frac{ql_{AB}}{1-ql_{AB}} = x \geq 1$, an AB type will always vote for A.

For the remaining statements, we consider two cases:

Case 1: Let $\pi_j > 0, j \in \{A,B,C\}$.

ii) Let $(\pi_A,\pi_C) \in E_1$.
Let $ql_{AB} < \frac{1}{3}$. From Lemma 2, $\lim_{O\to\infty}\frac{1}{O}\ln p_{jk}^N = \sup_{(a',c')\in E_{jk}} -I(a',c')$.

Note that $-I(.,.)$ is strictly concave and continuously differentiable in $int\Delta$, which implies that its level curves are closed and smooth. From lower semi-continuity of I, $\sup_{(a',c')\in E_{jk}} -I(a',c') = \max_{(a',c')\in closure(E_{jk})} -I(a',c')$. From strict concavity and differentiability - since these conditions imply smooth closed level curves of $-I$ -, the problems

$$\max_{(a',c')\in closure(E_{CB})} -I(a',c') \text{ and } \max_{(a',c')\in closure(E_{AB})} -I(a',c')$$

both admit unique optimizers in the statistically nearest points to $-I$, and we can solve:

$$\max_{(a',c')\in closure(E_{CB})} -I(a',c') \Leftrightarrow \begin{cases} \max -I(a',c') \\ s.t. c' = \frac{1}{2} - \frac{a'}{2} \\ 0 \leq a' \leq \frac{1-2x}{3} \end{cases} \tag{10}$$

[14] The properties verified are precisely the ones that ensure that (P_N) is a sequence of probability measures that satisfy the Large Deviation Principle (LDP) on χ with rate N and rate function I.

$$\max_{(a',c')\in closure(E_{AB})} - I(a',c') \Leftrightarrow \left\{ \begin{array}{c} \max -I(a',c') \\ s.t.c' = 1 - x - 2a' \\ \frac{1-2x}{3} \leq a' \leq \frac{1-x}{2} \end{array} \right. \tag{11}$$

Note that $p^N_{AC} << p^N_{CB}$ and $p^N_{AC} << p^N_{AB}$, since all points in E_{AC} are statistically more distant from $(\pi_A, \pi_C) \in E_1$ than are the optimizers of 10 and of 11.

We now solve both maximization problems for values of $(\pi_A, \pi_C) \in E_1$, for which the unique $\arg\max_{(a',c')\in closure(E_{jk})} -I(a',c')$ is not any of the extreme points of the line segments. Note that the strict concavity of $-I$ and the smoothness of its closed level curves imply that these optimizers are the tangency points between a level curve of $-I$ and the line segments.

Trivially, if $(\pi_A, \pi_C) \in closure(E_{CB})$, then $\lim_{O\to\infty} \frac{1}{O} \ln p^N_{CB} = 0$ and an AB type votes B, and if $(\pi_A, \pi_C) \in closure(E_{AB})$, then $\lim_{O\to\infty} \frac{1}{O} \ln p^N_{AB} = 0$, and an AB type votes A.

We now solve the generic problem

$$\left\{ \begin{array}{c} \min I(a',c') \\ s.t.c' = ma' + y \end{array} \right. \tag{12}$$

The Lagrangean is $L(a',c',\mu) = a' \ln \frac{a'}{\pi_A} + (1-a'-c') \ln \frac{(1-a'-c')}{\pi_B} + c' \ln \frac{c'}{\pi_C} + \mu(ma' + y - c')$, and the first order conditions (FOCs) are

$$\left\{ \begin{array}{c} \frac{\partial L}{\partial a'} = \ln a' + 1 - \ln \pi_A - \ln(1-a'-c') - 1 + \ln \pi_B + m\mu = 0 \\ \frac{\partial L}{\partial c'} = -\ln(1-a'-c') - 1 + \ln \pi_B + \ln c' + 1 - \ln \pi_C - \mu = 0 \\ \frac{\partial L}{\partial \mu} = ma' + y - c' = 0 \end{array} \right. \tag{13}$$

The solution (a^*, c^*, μ^*) is given by the following equivalent conditions

$$\left\{ \begin{array}{c} \frac{(ma^*+y)^m a^*}{(1-y-(1+m)a^*)^{m+1}} = \frac{\pi_A \pi_C^m}{\pi_B^{1+m}} \\ c^* = ma^* + y \\ \mu^* = \ln\left(\frac{c^* \pi_B}{(1-a^*-c^*)\pi_C}\right) \end{array} \right. \tag{14}$$

and

$$I(a^*, c^*) = \ln\left[\left(\frac{1-a^*-c^*}{\pi_B}\right)^{1-y} \left(\frac{c^*}{\pi_C}\right)^y\right] = \ln[(\frac{a^*}{\pi_A})^{(\frac{1-y}{1+m})}(\frac{c^*}{\pi_C})^{(\frac{m+y}{1+m})}] \tag{15}$$

Replacing $m = -\frac{1}{2}$ and $y = \frac{1}{2}$, the solution for 10 is given by

$$\begin{aligned} a^* &= \frac{\pi_A}{\pi_A + 2\sqrt{\pi_B}\sqrt{\pi_C}} \qquad (16) \\ c^* &= \frac{\sqrt{\pi_B}\sqrt{\pi_C}}{\pi_A + 2\sqrt{\pi_B}\sqrt{\pi_C}} \\ I_{CB}(a^*, c^*) &= \ln \frac{1}{\pi_A + 2\sqrt{\pi_B \pi_C}} \end{aligned}$$

Replacing $m = -2$ and $y = 1-x$, the solution for 11 is given by

$$\begin{aligned}
a^* &= \frac{1}{\pi_C^2 - 4\pi_A\pi_B}\left(\frac{1}{2}\pi_C\sqrt{x^2\pi_C^2 + 4\pi_A\pi_B - 4x^2\pi_A\pi_B} - \right. \\
&\left. -2\pi_A\pi_B - \frac{1}{2}x\pi_C^2 + 2x\pi_A\pi_B\right) \\
c^* &= \frac{\pi_C}{\pi_C^2 - 4\pi_A\pi_B}\left(\pi_C - \sqrt{x^2\pi_C^2 - 4\pi_A\pi_B x^2 + 4\pi_A\pi_B}\right) \\
I_{AB}(a^*, c^*) &= \ln\left(\frac{1}{\pi_C^2 - 4\pi_A\pi_B}\left(\pi_C - \sqrt{x^2\pi_C^2 - 4\pi_A\pi_B x^2 + 4\pi_A\pi_B}\right)\right) \times \\
&\times \left(\frac{1}{2\pi_B - 2x\pi_B}\left(x\pi_C + \sqrt{x^2\pi_C^2 - 4\pi_A\pi_B x^2 + 4\pi_A\pi_B}\right)\right)^x
\end{aligned} \tag{17}$$

From 2, an AB type will vote for A as long as $\lambda_{AB} \leq \overline{\lambda}_{AB} = \lim\limits_{N\to\infty} \frac{1+\frac{p_{AC}^N}{p_{AB}^N}}{1+\frac{p_{CB}^N}{p_{AB}^N}}$.

Now, if $I_{CB}(a^*, c^*) < I_{AB}(a^*, c^*)$, $\lim\limits_{N\to\infty} \frac{p_{CB}^N}{p_{AB}^N} = \infty$ and $\overline{\lambda}_{AB} = 0$, and if $I_{CB}(a^*, c^*) > I_{AB}(a^*, c^*)$, $\lim\limits_{N\to\infty} \frac{p_{CB}^N}{p_{AB}^N} = 0$ and $\overline{\lambda}_{AB} = 1$, since $\lim\limits_{N\to\infty} \frac{p_{AC}^N}{p_{AB}^N} = 0$ - any point in $closure(E_{AC})$ is statistically more distant than $\arg\max\limits_{(a',c')\in closure(E_{AB})} -I(a', c')$, for all $(\pi_A, \pi_C) \in E_1 \cap int\Delta$. Solving $I_{CB}(a^*, c^*) = I_{AB}(a^*, c^*)$ yields

$$\begin{aligned}
\frac{1}{\pi_A + 2\sqrt{\pi_B\pi_C}} &= \left(\frac{1}{\pi_C^2 - 4\pi_A\pi_B}\left(\pi_C - \sqrt{\overline{x}^2\pi_C^2 - 4\pi_A\pi_B\overline{x}^2 + 4\pi_A\pi_B}\right)\right) \times \\
&\times \left(\frac{1}{2\pi_B - 2\overline{x}\pi_B}\left(\overline{x}\pi_C + \sqrt{\overline{x}^2\pi_C^2 - 4\pi_A\pi_B\overline{x}^2 + 4\pi_A\pi_B}\right)\right)^{\overline{x}} \\
&\Leftrightarrow \ln\frac{\pi_C + \sqrt{\overline{x}^2\pi_C^2 - 4\pi_A\pi_B\overline{x}^2 + 4\pi_A\pi_B}}{\left(1-\overline{x}^2\right)\left(\pi_A + 2\sqrt{\pi_B\pi_C}\right)} \\
&= \overline{x}\ln\frac{1}{2\pi_B - 2\overline{x}\pi_B}\left(\overline{x}\pi_C + \sqrt{\overline{x}^2\pi_C^2 - 4\pi_A\pi_B\overline{x}^2 + 4\pi_A\pi_B}\right)
\end{aligned}$$

From strict convexity of I, and from the fact that I has closed and smooth level-curves, $I_{AB}(a^*, c^*)$ increases in $1-x$, that is, decreases in $x = \frac{ql_{AB}}{1-ql_{AB}}$, which is strictly increasing in q. Therefore, $I_{AB}(a^*, c^*)$ decreases in q.

Hence, $\lim\limits_{N\to\infty} \frac{p_{CB}^N}{p_{AB}^N} = \begin{cases} \infty, q < \overline{q} \\ 0, q > \overline{q} \end{cases}$, where $\overline{q}$ is given by 7, and we obtain the cutoff

$$\overline{\lambda}_{AB} = \lim_{N\to\infty} \frac{1+\frac{p_{AC}^N}{p_{AB}^N}}{1+\frac{p_{CB}^N}{p_{AB}^N}} = \begin{cases} 0, q < \overline{q} \\ 1, q > \overline{q} \end{cases}.$$

Hence, an AB type will vote for A if $q > \overline{q}$ and for B if $q < \overline{q}$.

If $ql_{AB} \in \left[\frac{1}{3}, \frac{1}{2}\right)$, $E_1 = \{(\pi_A, \pi_C) \in \Delta : \pi_B \geq x + \pi_A\}$ and in any level curve tangent to a point in $c' = 1 - x - a'$, I is smaller than $I_{CB}(a^*, c^*)$. Thus, $I_{CB}(a^*, c^*) > I_{AB}(a^*, c^*)$ and an AB type will vote for A.

iii) Let $(\pi_A, \pi_C) \in E_2$.

Let $ql_{AB} < \frac{1}{2}$. The proof goes exactly as in *ii*), only now $\lim_{N\to\infty} \frac{p_{AB}^N}{p_{CB}^N} = 0$, since any point in $closure(E_{AB})$ is more distant from $(\pi_A, \pi_C) \in E_2$ than $\arg\min_{(a',c')\in closure(E_{CB})} I(a', c')$. Also, if $(\pi_A, \pi_C) \in closure(E_{CB})$, then $\lim_{O\to\infty} \frac{1}{O} \ln p_{CB}^N = 0$ and an AB type votes B, and if $(\pi_A, \pi_C) \in closure(E_{AC})$, then $\lim_{O\to\infty} \frac{1}{O} \ln p_{AC}^N = 0$, and an AB type votes A. We now solve

$$\max_{(a',c')\in closure(E_{CB})} - I(a', c') \Leftrightarrow \begin{cases} \max -I(a', c') \\ s.t. c' = \frac{1+x}{2} - \frac{a'}{2} \\ 0 \leq a' \leq \frac{1-x}{3} \end{cases} \tag{18}$$

$$\max_{(a',c')\in closure(E_{AC})} - I(a', c') \Leftrightarrow \begin{cases} \max -I(a', c') \\ s.t. c' = x + a' \\ \frac{1-x}{3} \leq a' \leq \frac{1-x}{2} \end{cases} \tag{19}$$

Replacing $m = -\frac{1}{2}$ and $y = \frac{1+x}{2}$ in 14, the solution for 18 yields

$$\begin{aligned} a^* &= \pi_A \frac{\pi_A - \sqrt{x^2\pi_A^2 + 4\pi_B\pi_C - 4x^2\pi_B\pi_C}}{\pi_A^2 - 4\pi_B\pi_C} \\ c^* &= \frac{1}{2} \frac{\pi_A\sqrt{x^2\pi_A^2 + 4\pi_B\pi_C - 4x^2\pi_B\pi_C} - 4\pi_B\pi_C + x\pi_A^2 - 4x\pi_B\pi_C}{\pi_A^2 - 4\pi_B\pi_C} \\ I_{CB}(a^*, c^*) &= \ln\left(\frac{1}{\pi_A^2 - 4\pi_B\pi_C}\left(\pi_A - \sqrt{x^2\pi_A^2 - 4\pi_B\pi_C x^2 + 4\pi_B\pi_C}\right)\right) \times \\ &\times \left(\frac{x\pi_A + \sqrt{x^2\pi_A^2 + 4\pi_B\pi_C - 4x^2\pi_B\pi_C}}{2\pi_C(1-x)}\right)^x \end{aligned} \tag{20}$$

Replacing $m = 1$ and $y = x$ in 14, the solution for 19 yields

$$\begin{aligned} a^* &= \frac{1}{\pi_B^2 - 4\pi_A\pi_C}\left(\frac{1}{2}\pi_B\sqrt{x^2\pi_B^2 + 4\pi_A\pi_C - 4x^2\pi_A\pi_C} - \right. \\ &\left. -2\pi_A\pi_C - \frac{1}{2}x\pi_B^2 + 2x\pi_A\pi_C\right) \\ c^* &= \frac{1}{2} \frac{\pi_B\sqrt{x^2\pi_B^2 + 4\pi_A\pi_C - 4x^2\pi_A\pi_C} - 4\pi_A\pi_C + x\pi_B^2 - 4x\pi_A\pi_C}{\pi_B^2 - 4\pi_A\pi_C} \\ I_{AC}(a^*, c^*) &= \ln\left(\frac{1}{\pi_B^2 - 4\pi_A\pi_C}\left(\pi_B - \sqrt{x^2\pi_B^2 - 4\pi_A\pi_C x^2 + 4\pi_A\pi_C}\right)\right) \times \\ &\times \left(\frac{x\pi_B + \sqrt{x^2\pi_B^2 - 4\pi_A\pi_C x^2 + 4\pi_A\pi_C}}{2\pi_C(1-x)}\right)^x \end{aligned} \tag{21}$$

Due to the strict convexity of I, and to the smoothness of its closed level curves, $I_{CB}(a^*, c^*) = I_{AC}(a^*, c^*)$ has a unique solution. Since $I_{CB}(a^*, c^*)$ and $I_{AC}(a^*, c^*)$ are symmetric in the exchange of π_B and π_A, $I_{CB}(a^*, c^*) = I_{AC}(a^*, c^*) \Rightarrow \pi_B = \pi_A$[15]. Hence,

$$\overline{\lambda}_{AB} = \lim_{N\to\infty} \frac{\frac{p^N_{AB}}{p^N_{CB}} + \frac{p^N_{AC}}{p^N_{CB}}}{\frac{p^N_{AB}}{p^N_{CB}} + 1} = \begin{cases} \infty, \pi_A > \pi_B \\ 0, \pi_A < \pi_B \end{cases}$$

and an AB type votes A if $\pi_A > \pi_B$ and votes B if $\pi_A < \pi_B$.

iv) Let $(\pi_A, \pi_C) \in E_3$[16]. Then,

$$\overline{\lambda}_{AB} = \lim_{N\to\infty} \frac{1 + \frac{p^N_{AC}}{p^N_{AB}}}{1 + \frac{p^N_{CB}}{p^N_{AB}}} \geq 1, \text{ since} \lim_{N\to\infty} \frac{p^N_{CB}}{p^N_{AB}} = 0$$

.

Note that $\min_{(a',c')\in closure(E_{CB})} I(a', c') > \min_{(a',c')\in closure(E_{AB})} I(a', c')$.

Case 2: In order to complete the proof, we must establish the RBR in case any π_j is zero.

ii) Let $\pi_A = 0$. Then, $I(c') = (1-c') \ln \frac{(1-c')}{\pi_B} + c' \ln \frac{c'}{\pi_C}$ is the rate function of the binomial distribution. E_1 simplifies to $E_1 = \left\{\pi_C : \pi_C \leq \frac{1}{2} \wedge \pi_C \leq 1 - x \wedge x < 1\right\}$ and problems 10 and 11 respectively reduce to $\begin{cases} \max -I(c') \\ s.t. c' = \frac{1}{2} \end{cases}$ and $\begin{cases} \max -I(c') \\ s.t. c' = 1 - x \end{cases}$. If $1 - x > \frac{1}{2} \Leftrightarrow x < \frac{1}{2}$, then $I(\frac{1}{2}) < I(1 - x)$ and an AB type votes for B. Conversely, if $x > \frac{1}{2}$, an AB type votes for A.

Let $\pi_C = 0$. An AB type votes A: C gets zero votes and voting for B would be a dominated action.

iii) Let $\pi_A = 0$. $E_2 = \left\{\pi_C : \pi_C \geq \frac{1+x}{2} \wedge \pi_C \geq x \wedge x < 1\right\}$ and problems 18 and 19 respectively reduce to $\begin{cases} \max -I(c') \\ s.t. c' = \frac{1+x}{2} \end{cases}$ and $\begin{cases} \max -I(c') \\ s.t. c' = x \end{cases}$. If $\frac{1+x}{2} > x \Leftrightarrow x < 1$, then $I(\frac{1+x}{2}) < I(x)$ and an AB type votes for B. As established in *i*), if $x > 1$, an AB type votes for A.

Let $\pi_B = 0$. Then an AB type votes A: since $a \geq b = 0$, if by voting for A an AB type can not ensure that A beats C, then B will also lose to C with her votes, and voting for B would be a dominated action.

v) If $\pi_A = \pi_C = 0 \Rightarrow c = 0$, $p^N_{CB} = 0$ and from Lemma 1 an AB type votes A.

If $\pi_A = 0$ and $\pi_C = 1 \Rightarrow c = O$, $b = 0$, $p^N_{CB} = 0$, since $b = 0 \geq c - M > 0$ is impossible, and from Lemma 1 an AB type votes A.

[15] Note that Lagrange multipliers are $\mu^*_{CB} = \mu^*_{AC} =$
$= \ln \pi_B \frac{\pi_A\sqrt{x^2\pi_A^2+4\pi_B\pi_C-4x^2\pi_B\pi_C}-4\pi_B\pi_C+x\pi_A^2-4x\pi_B\pi_C}{\pi_C\left(\pi_A\sqrt{x^2\pi_A^2+4\pi_B\pi_C-4x^2\pi_B\pi_C}-4\pi_B\pi_C-x\pi_A^2+4x\pi_B\pi_C\right)}$.

[16] Note that $\pi_A > 0$.

This completes the proof. ■