SCATTERING RESULTS FOR DISPERSIVE SYSTEMS

Functional Analysis and Applications Seminar - Universidade de Aveiro

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The Schrödinger-Debye system describes the propagation of an electromagnetic wave through a medium whose response cannot be considered instantaneous:

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- $v : (x,t) \in \mathbb{R}^d \times \mathbb{R} \to v(x,t) \in \mathbb{R}$;
- $\mu > 0$;
- $\lambda = 1$ (defocusing) or $\lambda = -1$ (focusing).

This last terminology is inherited from the Cubic Schrödinger Equation ($\mu = 0$).

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Strichartz Estimates

$$\|S(t)\phi\|_{L_t^p L_x^q} \lesssim \|\phi\|_{L^2}$$

$$\left\| \int_0^t S(t-s) f(x,s) ds \right\|_{L_t^p L_x^q} \lesssim \|f\|_{L_t^{p'} L_x^{q'}}$$

for (p,q) admissible, that is

$$\frac{2}{p} = d\Big(\frac{1}{2} - \frac{1}{q}\Big) \text{ and } 2 \leq q \leq \frac{2d}{d-2}.$$

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$$\frac{d}{dt}E(t) = 2\lambda\mu \int v_t^2,$$

where

$$E(t) = \int |\nabla u|^2 + 2v|u|^2 - \lambda v^2.$$

Adán Corcho, Jorge D. Silva & FO

Proceedings of the AMS, vol. 141, pp 3485 - 3499, 2013.

Theorem

Let $(u_0, v_0) \in H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ and $\lambda = \pm 1$. Then, for all $T > 0$, there exists a unique solution

$$(u, v) \in C([0; T], H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2))$$

to the Initial Value Problem associated to the Schrödinger-Debye system.

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In other words, the nonlinear solution u behaves as the linear solution $S(t)u_+$ for large times.

(Simão Correia & FO, Nonlinearity, Vol 31, 7, 2018)
Scattering of small solutions in dimension $d = 4$
Let

$$(X, Y) \in \{(L^2(\mathbb{R}^4), L^2(\mathbb{R}^4)), (H^1(\mathbb{R}^4), H^1(\mathbb{R}^4)), (\Sigma(\mathbb{R}^4), H^1(\mathbb{R}^4))\}.$$

There exists $\epsilon > 0$ such that, if $(u_0, v_0) \in X \times Y$ satisfies $\|u_0\|_X + \|v_0\|_Y < \epsilon$, then the corresponding solution (u, v) of the Schrödinger-Debye system is global and scatters, that is, there exists $u_+ \in X$ such that

$$\|u(t) - S(t)u_+\|_X \to 0 \text{ and } \|v(t)\|_Y \to 0, \quad t \to \infty. \tag{1}$$

In the particular case $(X, Y) = (\Sigma(\mathbb{R}^4), H^1(\mathbb{R}^4))$, the following decay estimate holds:

$$\|u(t)\|_{L^p(\mathbb{R}^4)} \lesssim \frac{C(\|u_0\|_{\Sigma(\mathbb{R}^4)}, \|v_0\|_{H^1(\mathbb{R}^4)})}{t^{\left(2-\frac{4}{p}\right)}}, \quad t > 0,\ 2 < p < 4. \tag{2}$$

(Simão Correia & FO, Nonlinearity, Vol 31, 7, 2018)

Scattering of small solutions in dimensions $d = 2, 3$

There exists $\delta > 0$ such that, if $(u_0, v_0) \in \Sigma(\mathbb{R}^d) \times H^1(\mathbb{R}^d)$, $d = 2, 3$, satisfies $\|u_0\|_{H^1(\mathbb{R}^d)} + \|v_0\|_{H^1(\mathbb{R}^d)} < \delta$, then the corresponding solution (u, v) of the Schrödinger-Debye System scatters, that is, there exists $u_+ \in \Sigma(\mathbb{R}^d)$ such that

$$\|u(t) - S(t)u_+\|_{\Sigma(\mathbb{R}^d)} \to 0,\ \|v(t)\|_{H^1(\mathbb{R}^d)} \to 0, \quad t \to \infty.$$

Furthermore,

$$\|u(t)\|_{L^p(\mathbb{R}^d)} \lesssim \frac{C(\|u_0\|_{\Sigma(\mathbb{R}^d)}, \|v_0\|_{H^1(\mathbb{R}^d)})}{t^{d\left(\frac{1}{2}-\frac{1}{p}\right)}}, \quad t > 0,\ 2 < p < 2d/(d-2)^+. \quad (3)$$

(Simão Correia & FO, Nonlinearity, Vol 31, 7, 2018)

Modified Scattering in dimension $d=1$

There exists $\epsilon>0$ such that, if $(u_0,v_0)\in\Sigma(\mathbb{R})\times H^1(\mathbb{R})$ satisfies $\|u_0\|_{H^1(\mathbb{R})}+\|v_0\|_{H^1(\mathbb{R})}<\epsilon$, then the corresponding solution (u,v) of the Schrödinger-Debye system scatters up to a phase correction, that is, there exists (a unique) $u_+\in L^2(\mathbb{R})$ such that

$$\|e^{i\Psi(t)}\widehat{S(-t)u}(t)-\widehat{u_+}\|_{L^2(\mathbb{R})}\to 0,\ \|v(t)\|_{L^\infty(\mathbb{R})}\to 0,\quad t\to\infty,$$

where $\Psi(\xi,t)=\displaystyle\int_1^t\int_1^s\frac{1}{2s'}e^{-(s-s')}\left|\widehat{f}\left(\frac{s}{s'}\xi,s'\right)\right|^2ds'ds$ and $f=S(-t)u$.
Also,

$$\|u(t)\|_{L^\infty(\mathbb{R})}\lesssim\frac{1}{t^{\frac12}},\quad t\to+\infty.$$

We begin by the global well-posedness of solutions for small initial data $(u_0, v_0) \in L^2(\mathbb{R}^4) \times L^2(\mathbb{R}^4)$.

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From the local well-posedness theory, we get the following blow-up alternative:
If $[0; T^*[$ is the maximal time interval of existence, $\lim\limits_{t \to T^*} h(t) = +\infty$, where

$$h(t) = \|u\|_{L^\infty_T L^2_x} + \|u\|_{L^2_T L^4_x}.$$

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Integral formulation:

$$v(t) = e^{-t/\mu} v_0 + \frac{\lambda}{\mu} \int_0^t e^{-(t-s)/\mu} |u(s)|^2 ds,$$

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that is,

$$u(t) = S(t)u_0 + i \int_0^t S(t-s)\Big(e^{-s/\mu} v_0 + \frac{\lambda}{\mu} \int_0^s e^{-(s-s')/\mu} |u(s')|^2 ds'\Big) u(s)ds.$$

We set $f(t) = S(-t)u(t)$. Since

$$\|u\|_{L^2((0,\infty);L^4_x)} < \infty,$$

we have

$$\begin{aligned}\|f(t) - f(t')\|_{L^2} &= \|S(t)(f(t) - f(t'))\|_{L^2} \\ &\lesssim \|u\|_{L^2((t',t);L^4_x)}\|v_0\|_{L^2} + \|u\|^3_{L^2((t',t);L^4_x)} \to 0, \quad t, t' \to \infty\end{aligned}$$

Hence there exists $u_+ := \lim_{t\to\infty} S(-t)u(t) \in L^2(\mathbb{R}^4)$.