Ground states for a Schrödinger System

arizing in nonlinear optics

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Third harmonic generation

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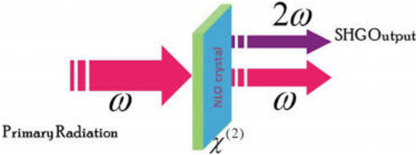

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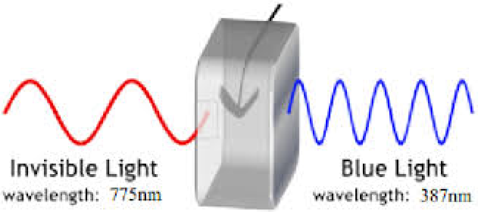

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From the Maxwell-Faraday's equation $\dfrac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E}$

and Ampère's Law $\vec{\nabla} \times \vec{B} = \mu_0 \dfrac{\partial \vec{D}}{\partial t}$,

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} + \mu_0 \frac{\partial^2 \vec{D}}{\partial t^2} = 0.$$

Third harmonic generation

Using the constitutive law

$$\vec{D} = \mathrm{n}^2 \epsilon_0 \vec{E} + 4\pi \epsilon_0 \vec{P}_{NL},$$

where $\vec{P}_{NL}$ is the nonlinear part of the polarization vector and n the linear refractive index, the identity $\mu_0 \epsilon_0 c^2 = 1$ and noticing that

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\Delta \vec{E} + \vec{\nabla}(\vec{\nabla} \cdot \vec{E}),$$

we get, after neglecting the last term in this identity, the vectorial wave equation

$$\Delta \vec{E} - \frac{\mathrm{n}^2}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = \frac{4\pi}{c^2} \frac{\partial^2 \vec{P}_{NL}}{\partial t^2}, \tag{1}$$

Third harmonic generation

Assuming that the beams propagate in a slab waveguide, in the direction of the (Oz) axis, we decompose one of the transverse directions of $\vec{E}$ in two frequency components as

$$E = \Re e\left(E_1 e^{i(k_1 z - \omega t)} + E_3 e^{i(k_3 z - 3\omega t)}\right).$$

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Inserting in (1), with $P_{NL} = \chi^{(3)} E^3$,

$$\begin{cases} \Delta_\perp E_1 + 2ik_1 \dfrac{\partial E_1}{\partial z} + \Big(\dfrac{(\mathrm{n}(\omega))^2\omega^2}{c^2} - k_1^2\Big) E_1 + \chi(|E_1|^2 E_1 + 2|E_3|^2 E_1 + E_3 \overline{E}_1^2 e^{-i(3k_1 - k_3)z}) = 0 \\ \Delta_\perp E_3 + 2ik_3 \dfrac{\partial E_3}{\partial z} + \Big(\dfrac{9(\mathrm{n}(3\omega))^2\omega^2}{c^2} - k_3^2\Big) E_3 + 9\chi(2|E_1|^2 E_3 + |E_3|^2 E_3 + \dfrac{1}{3} E_1^3 e^{-i(3k_1 - k_3)z}) = 0, \end{cases}$$

Third harmonic generation

Rescaling $(E_1, E_3) \to (u, w)$, and for $\sigma = k_3/k_1$,
$\mu = 3(k_3 - 3k_1 + \sigma)$,

$$\begin{cases} iu_t + \Delta u - u + (\frac{1}{9}|u|^2 + 2|w|^2)u + \frac{1}{3}\overline{u}^2 w = 0, \\ i\sigma w_t + \Delta w - \mu w + (9|w|^2 + 2|u|^2)w + \frac{1}{9}u^3 = 0, \end{cases}$$

where the z direction is now called t.

Notice that at resonance, $\sigma = 3$ and $\mu = 9$.

Hamiltonian structure

Nonlinear Schrödinger system with cubic nonlinearity

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Defining $U = (u, w)$, $J = \mathrm{diag}(\frac{1}{i}, \frac{1}{i\sigma})$ and

$$H(u, v) = \frac{1}{2}\int (|\nabla u|^2 + |\nabla v|^2 + |u|^2 + \mu|w|^2) \\ - \int \left(\frac{1}{36}|u|^4 + \frac{9}{4}|w|^4 + |u|^2|w|^2 + \frac{1}{9}\Re e(\overline{u}^3 w)\right),$$

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Hamiltonian structure and conservation of energy

$$JU_t = H'(U).$$

Conservation of mass and Hamiltonian invariance

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From this equality we can obtain

Conservation of mass

$$\frac{d}{dt}M(u, w) = 0,$$

where

$$M(u, v) = \frac{1}{2}\int |u|^2 + 3\sigma|w|^2.$$

Localized solutions and bound states

We look for solutions of the form

$$u(x,t) = e^{i\omega t}P(x), \quad w(x,t) = e^{3i\omega t}Q(x),$$

where P and Q are real functions with a suitable decay at ∞.

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These functions (bound states) satisfy

Bound States

$$\begin{cases} \Delta P - (\omega+1)P + \left(\frac{1}{9}P^2 + 2Q^2\right)P + \frac{1}{3}P^2 Q = 0, \\ \Delta Q - \Big(\mu + 3\sigma\omega)Q + (9Q^2 + 2P^2\Big)Q + \frac{1}{9}P^3 = 0. \end{cases}$$

Action and ground states

We define the action

$$S(P,Q) = E(P,Q) + \omega M(P,Q)$$

and single-out the set of ground states, minimizing the action among all bound states ($\mathcal{B}$):

$$\mathcal{G} = \{(P_0, Q_0) \in \mathcal{B}\,;\, \forall (P,Q) \in \mathcal{B},\ S(P_0,Q_0) \leq S(P,Q)\}.$$

Existence of Ground States

Theorem

Let $1 \leq n \leq 3$, $\sigma, \mu > 0$ *and* $\omega > \max\{-1, -\mu/3\sigma\}$. *Then the set of ground states,* $\mathcal{G}(\omega, \mu, \sigma)$ *is nonempty.*
In addition, there exists at least one ground state (P_0, Q_0) *which is radially symmetric,* Q_0 *is positive and* P_0 *is either positive or identically zero.*

Existence of Ground States - Strategy

We consider the set $\mathcal{N} = \{(u,v) \neq (0,0) \, : \, S'(u,v) \perp_{L^2} (u,v)\}$.
For $(u,w) \neq (0,0)$ is in $\mathcal{N}$ iff

$$\tau(u,w) := \int |\nabla u|^2 + |\nabla w|^2 + (1+\omega)u^2 + (\mu + 3\sigma\omega)w^2$$

$$-\frac{1}{9}u^4 - 4u^2w^2 - 9w^4 - \frac{4}{9}u^3 w = 0.$$

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In fact, $\mathcal{N}$ is a complete regular manifold: $(0,0)$ is an isolated point of the set $\{\tau = 0\}$ and $\langle \tau'(u,w),(u,w)\rangle \neq 0$ for all $(u,w) \in \mathcal{N}$.

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Furthermore, the minimizers of $inf_{\mathcal{N}} S$ are ground states:

Indeed, $S'(u_0,w_0) = \lambda \tau'(u_0,w_0) \Rightarrow \lambda = 0$.

Existence of Ground States - Strategy

The (simplified steps are the following:)

- We consider a minimizing sequence $(u_n, w_n) \in \mathcal{N}$;
- We take the Schwarz symmetization (u_n^*, w_n^*) and project it in $\mathcal{N}$: for some t, $(tu_n^*, tw_n^*) \in \mathcal{N}$;
- We show that it is still a minimizing sequence;
- We use the compact injection

$$H^1_{rd}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)\, p > 2,$$

to obtain a minimizer.

Semitrivial vs Nontrivial Ground States

Theorem

In addition to the assumptions of the existence Theorem:

- *If* $\mu = 3\sigma$ *and* $\mu \geq 9^{\frac{4}{4-n}}$*:*
  All ground states are nontrivial: $P \neq 0$ *and* $Q \neq 0$.
- *if* $\omega + 1 = \mu + 3\sigma\omega$*:*
  All ground states of the form $(0, Q)$ *and* Q *is a ground state of*

$$\Delta Q - (\mu + 3\sigma\omega)Q + 9Q^3 = 0.$$

In particular, up to translation, ground states are unique.

Fully non-trivial Ground States

Let

$$N(u,w) := \int \left(\frac{1}{36} u^4 + \frac{9}{4} w^4 + u^2 w^2 + \frac{1}{9} u^3 w \right).$$

and

$$K(u,w) = \|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2.$$

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We prove the existence of $\theta, t \in \mathbb{R}$ and $W \in H^1$ such that $(t\theta W, tQ) \in \mathcal{N}$ and $S(t\theta W, tQ) < S(0, Q)$.

Fully non-trivial Ground States

- $t^2 = \dfrac{K(\theta W, Q) + (1+\omega) M(\theta W, Q)}{4N(\theta W, Q)}$ assures that $(t\theta W, tQ) \in \mathcal{N}$;
- $S(t\theta W, tQ) < S(0, Q)$ if and only if

$$\Big(K(\theta W, Q) + (1+\omega) M(\theta W, Q)\Big)^2$$
$$< 4N(\theta W, Q)\Big(K(0, Q) + (\omega+1) M(0, Q)\Big).$$

- Coefficients of θ^4:

$$(K(W, 0) + (\omega+1) M(W, 0))^2$$
$$< \frac{1}{9}\left(\int W^4\right)\Big(K(0, Q) + (\omega+1) M(0, Q)\Big).$$

Fully non-trivial Ground States

Setting $W(x) = Q(\lambda x)$, and using the homogeneity of the functionals, the condition boils down to

$$f(\lambda) = \frac{n\mu}{4-n}\lambda^2 + 1 - \frac{4\mu}{9(4-n)}\lambda^{n/2} < 0.$$

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f has a global minimum at $\lambda_0 = 9^{-2/(4-n)}$ and $f(\lambda_0) = 1 - \mu\lambda_0^2$.

Local Well-Posedness

Theorem

Let $1 \leq n \leq 3$ and $u_0, w_0 \in H^1(\mathbb{R}^n)$. Then, the Cauchy problem admits a unique solution,

$$U = (u, w) \in C((-T_*, T^*); H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n))$$

defined in the maximal interval of existence $(-T_*, T^*)$, where $T_*, T^* > 0$.

In addition, the following blow-up alternative holds: if $T^* < \infty$ then

$$\lim_{t \to T^*} (K(u, w)) = +\infty,$$

where

$$K(u, w) = \|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2.$$

Global Well-posedness

$$
\begin{aligned}
2K(u,w) = 2H_0 &- \int \left(|u|^2 + \mu|w|^2\right) \\
&- \int \left(\frac{1}{18}|u|^4 + \frac{9}{2}|w|^4 + |u|^2|w|^2 + \frac{2}{9}\Re e(\overline{u}^3 w) \right),
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$$K(U) \leq H_0 + CM_0^{2-\frac{n}{2}} K(U)^{\frac{n}{2}}.$$

Global Well-Posedness - subcritical case $n = 1$

$$K(U) \leq H_0 + CM_0^{\frac{3}{2}} K(U)^{\frac{1}{2}} \leq H_0 + C\left(\frac{1}{\epsilon} M_0^3 + \epsilon K(U)\right):$$

$$(1 - C\epsilon)K(U) \leq H_0 + \frac{C}{\epsilon} M_0^3.$$

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Theorem

For $n = 1$ and $(u_0, w_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ the Cauchy problem is globally well-posed.

Global Well-Posedness - critical case $n = 2$

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$$\int \left(\frac{1}{36}|u|^4 + \frac{9}{4}|w|^4 + |u|^2|w|^2 + \frac{1}{9}|u|^3|w| \right) \leq CK(u,w)M(u,w):$$

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$$\frac{1}{C} = \inf \left\{ J(u,w) := \frac{K(u,w)M(u,w)}{N(u,w)} \ : \ N(u,w) > 0 \right\}.$$

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In fact, this infimum is achieved at (any) ground state (P, Q) with $\mu = 3\sigma$ and $\omega = 0$.

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In fact, this infimum is achieved at (any) ground state (P, Q) with $\mu = 3\sigma$ and $\omega = 0$. Furthermore,

$$\frac{1}{C} = M(P, Q).$$

Global Well-Posedness - critical case $n = 2$

Theorem

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Theorem

Assume $M(u_0, w_0) < M(P, Q)$. Then the Cauchy problem is globally well-posed.

This condition is sharp, at least at resonance.

Global Well-Posedness - supercritical case $n = 3$

We have

$$K(U(t)) \leq H_0 + CM_0^{\frac{1}{2}} K(U(t))^{\frac{3}{2}}.$$

$$f(K(U)) \geq 0 \; where \; f(r) = H_0 - r + CM_0^{\frac{1}{2}} r^{\frac{3}{2}}.$$

Global Well-Posedness - supercritical case $n = 3$

We can prove:

Theorem

Assume $n = 3$ and $u_0, w_0 \in H^1(\mathbb{R}^3)$. Suppose that

$$H(u_0, w_0)M(u_0, w_0) < \frac{1}{2}H(P, Q)M(P, Q)$$

and

$$K(u_0, w_0)M(u_0, w_0) < K(P, Q)M(P, Q),$$

where (P, Q) is any ground state with $\omega = 0$ and $\mu = 3\sigma$. Then, as long as the local solution given in exists, there holds

$$K(u(t), w(t))M(u(t), w(t)) < K(P, Q)M(P, Q).$$

In particular, this implies that the Cauchy problem is globally well-posed under these conditions.

Blow-up

Assume

$$u_0, w_0 \in \Sigma = H^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n, |x|^2 dx)$$

and define

$$V(t) = \int |x|^2(|u(t)|^2 + 3\sigma|w(t)|^2),$$

where $(u(t), w(t))$ is the maximal solution with initial data (u_0, w_0), and defined in the maximal time interval $[0, T^*)$.

Then $V \in C^2([0, T^*))$.

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If $V''(t) < 0$ for all t, the solution cannot exist globally in time.

Blow-up

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$$u_0, w_0 \in \Sigma = H^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n, |x|^2 dx)$$

and define

$$V(t) = \int |x|^2 (|u(t)|^2 + 3\sigma |w(t)|^2),$$

where $(u(t), w(t))$ *is the maximal solution with initial data* (u_0, w_0)*, defined in the maximal time interval* $[0, T^*)$*. Then* $V \in C^2([0, T^*))$*. In addition,*

$$V'(t) = 4Im \int (\overline{u}(t) x \cdot \nabla u(t) + 3\overline{w}(t) x \cdot \nabla w(t))$$

Blow up

Theorem

Furthermore,

$$V''(t) = \int \left(8|\nabla u|^2 + 8|\nabla w|^2 - \frac{2n}{9}|u|^4 - \frac{54n}{\sigma}|w|^4 - 8n|u|^2|w|^2 \right)$$

$$+2\left(\frac{24}{\sigma} - 8\right) \Re e \int \overline{u}|w|^2 x \cdot \nabla u + \frac{1}{9}\left(\frac{12}{\sigma} - 12\right) n \Re e \int \overline{u}^3 w$$

$$+\frac{1}{9}\left(\frac{24}{\sigma} - 8\right) \Re e \int 3\overline{u}^2 w x \cdot \nabla u.$$

For $\sigma = 3$ (at resonance),

$$V''(t) = 8nH(u_0, w_0) + 4(2-n)\int |\nabla u|^2 + |\nabla w|^2 - 4n \int |u|^2 + \mu |w|^2.$$

Blow-up

Theorem

Let $u_0, v_0 \in \Sigma := H^1 \cap L^2(|x|^2 dx)$ *and* $]-T_*, T^*[$ *the maximal time interval of existence of the solution given by the local-wellposedness result.*
For $n = 2, 3$, $\sigma = 3$ *and* $\mu = 9$, *if*

$$2H(u_0, w_0) < M(u_0, w_0)$$

then $T_* < +\infty$ *and* $T^* < +\infty$.

Blow-up

Theorem

Also,

(i) *If* $2E(u_0, w_0) = M(u_0, w_0)$ *and*

$$Im \int (\overline{u}_0 x \cdot \nabla u_0 + 3\overline{w}_0 x \cdot \nabla w_0) < 0,$$

then $T^* < \infty$.

(ii) *If* $2E(u_0, w_0) = M(u_0, w_0)$ *and*

$$Im \int (\overline{u}_0 x \cdot \nabla u_0 + 3\overline{w}_0 x \cdot \nabla w_0) > 0,$$

then $T_* < \infty$.

Blow-up

Theorem

(iii) *If* $2H(u_0, w_0) > M(u_0, w_0)$ *and*

$$\sqrt{2} Im \int (\overline{u}_0 x \cdot \nabla u_0 + 3\overline{w}_0 x \cdot \nabla w_0)$$

$$< -\sqrt{n(2E(u_0, w_0) - M(u_0, w_0))M(xu_0, xw_0)}$$

then $T^* < \infty$.

(iv) *If* $2H(u_0, w_0) > M(u_0, w_0)$ *and*

$$\sqrt{2} Im \int (\overline{u}_0 x \cdot \nabla u_0 + 3\overline{w}_0 x \cdot \nabla w_0)$$

$$> \sqrt{n(2E(u_0, w_0) - M(u_0, w_0))M(xu_0, xw_0)}$$

then $T_* < \infty$.

Blow-up

Theorem

Assume $n = 3$, $\sigma = 3$, $\mu = 9$. *Suppose that* $u_0, w_0 \in \Sigma$ *and*

$$H(u_0, w_0)M(u_0, w_0) < \frac{1}{2}H(P, Q)M(P, Q) \tag{2}$$

and

$$K(u_0, w_0)M(u_0, w_0) > K(P, Q)M(P, Q), \tag{3}$$

where (P, Q) *is any ground state with* $\omega = 0$ *(and* $\mu = 3\sigma$*). Then the solution blows up in finite time.*

(In)stability of Ground States $(e^{i\omega t}P(x), e^{3i\omega t}Q(x))$

Recall that the system is invariant by translations and rotations:

If (u, w) is a solution so are

$$(u(\cdot + y)w(\cdot + y)) \text{ and } (e^{i\theta}u, e^{3i\theta}w).$$

We introduce the orbit generated by (P, Q) is defined by

$$\mathcal{O}_{P,Q} = \{(e^{i\theta}P(\cdot + y), e^{3i\theta}Q(\cdot + y)) : \quad \theta \in \mathbb{R}, y \in \mathbb{R}^n\}.$$

(In)stability of Ground States $(e^{i\omega t}P(x), e^{3i\omega t}Q(x))$

Definition (Orbital stability)

We say that a standing wave $(e^{i\omega t}P, e^{3i\omega t}Q)$ is orbitally stable if for any $\epsilon > 0$ there exists a $\delta > 0$ with the following property: if $(u_0, w_0) \in H^1 \times H^1$ satisfies $\|(u_0, w_0) - (P, Q)\|_{H^1 \times H^1} < \delta$ then the solution with initial data (u_0, w_0) is global and satisfies

$$\sup_{t\in\mathbb{R}} \inf_{(\theta,y)\in\mathbb{R}\times\mathbb{R}^n} \|(u(t), w(t)) - (e^{i\theta}u(\cdot + y), e^{3i\theta}u(\cdot + y))\|_{H^1\times H^1} < \epsilon.$$

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$$\sup_{t \in \mathbb{R}} \inf_{(\theta, y) \in \mathbb{R} \times \mathbb{R}^n} \|(u(t), w(t)) - (e^{i\theta}u(\cdot + y), e^{3i\theta}u(\cdot + y))\|_{H^1 \times H^1} < \epsilon.$$

- Strong instability: There exists initial data arbitrary close to (P, Q) such that the corresponding solution blows-up in finite time.
- Weak instability: Given any neighbourhood $\mathcal{O}^{(\epsilon)}_{(P,Q)}$ of $\mathcal{O}_{(P,Q)}$ there exists initial data arbitrary close to (P, Q) such that the corresponding solution leaves $\mathcal{O}^{(\epsilon)}_{(P,Q)}$ in finite time.

Instability of Ground States $(e^{i\omega t}P(x), e^{3i\omega t}Q(x))$

Theorem

Assume either $n = 3$ and $\mu > 0$ or $n = 2$ and $\mu \neq 3\sigma$. Then all real ground states (P, Q) are weakly orbitally unstable.

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Let

$$\Sigma := \left\{(u, w) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) : \; M(u, w) = M(P, Q)\right\}.$$

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General criterium: The existence of Ψ such that

(i) Ψ belongs to the tangent space $T_{(P,Q)}\Sigma$;

(ii) $\langle S''(P, Q)\Psi, \Psi\rangle < 0$;

(iii) $+$ some geometric (straightforward) conditions.

(In)stability of Ground States $(e^{i\omega t}P(x), e^{3i\omega t}Q(x))$

We will take $\Psi = \Gamma'(0)$ with

$$\Gamma(t) = \left(\gamma(t)\lambda^{\frac{n}{2}}(t)P(\lambda(t)\cdot), \alpha(t)\lambda^{\frac{n}{2}}(t)Q(\lambda(t)\cdot)\right),$$

where α, γ, and λ are smooth functions to be chosen later satisfying,

$$\alpha(0) = \gamma(0) = \lambda(0) = 1$$

and, setting $k = \dfrac{\int P^2}{3\sigma \int Q^2}$,

$$\gamma^2 k + \alpha^2 = k + 1.$$

(In)stability of Ground States $(e^{i\omega t}P(x), e^{3i\omega t}Q(x))$

$S(\Gamma(t)) = E(\Gamma(t)) + \frac{\omega}{2}M(P,Q)$, because $\Gamma(t) \subset \Sigma$. Thus,

$$\frac{d^2}{dt^2}E(\Gamma(t)) = \frac{d^2}{dt^2}S(\Gamma(t)) = \langle S''(\Gamma(t))\Gamma'(t), \Gamma'(t)\rangle + \langle S'(\Gamma(t)), \Gamma''(t)\rangle.$$

Evaluating at $t = 0$ and using that $S'(P,Q) = 0$, we see that

$$\langle S''(P,Q)\Psi, \Psi\rangle < 0$$

is equivalent to

$$\frac{d^2}{dt^2}E(\Gamma(t))\Big|_{t=0} < 0.$$

We get

$$
\begin{aligned}
&\frac{d^2}{dt^2}E(\Gamma(t))\Big|_{t=0} = \\
&= \alpha_0^2\left[\int\left(-\frac{2}{k^2}P^4+\frac{8}{k}P^2Q^2-18Q^4+\left(\frac{2}{3k}+\frac{1}{9}-\frac{1}{3k^2}\right)P^3Q\right)\right] \\
&\quad +2\alpha_0\lambda_0\left[2(3\sigma-\mu)\int Q^2\right. \\
&+(n-2)\int\left(\frac{1}{9k}P^4-9Q^4+\left(\frac{2}{k}-2\right)P^2Q^2+\left(\frac{1}{3k}-\frac{1}{9}\right)P^3Q\right)\Bigg] \\
&\quad +\lambda_0^2\frac{n(2-n)}{4}\int\left(\frac{1}{9}P^4+9Q^4+4P^2Q^2+\frac{4}{9}P^3Q\right) \\
&\equiv A_0\alpha_0^2+2B_0\alpha_0\lambda_0+C_0\lambda_0^2.
\end{aligned}
$$