1) Alpha's production function is $f(L, K) = L^{0.5}K^{0.5}$. The prices of labour (L) and capital (K) are w= 5 and r= 2 respectively.
   a) The firm is now using 50 units of labour and 375 units of capital. Show that the firm is not operating efficiently. Should it use more labour and less capital or the opposite? Explain.
   b) How much labour and capital should the firm use to minimise the cost of the current level of output?
   c) Check that the output level is the same, and see how much the firm saves by switching to the efficient input combination.
   d) Suppose now the firm wants to produce as much as it can, but cannot spend more on inputs that was spending in part a). How much additional output can the firm achieve with the same total cost if it operates efficiently? Explain.

2) A firm's technology is described by the production function $f(L, K) = 4L^{0.25}K^{0.25}$.
   a) Find the conditional factor demand functions and explain what they are.
   b) Find the cost function. Explain what it is.
   c) What does the 4 in the production function mean?

3) A firm's technology is described by the production function $f(L, K) = aL^{1/2}K^{1/6}$
   a) Find the conditional factor demand functions.
   b) Find the cost function.
   c) Input prices are w = 6 and r = 2. Find the cost curve.
   d) We know one unit of output at the previous input prices cost 80. How much is a?

4) A firm operates with the production function $y = L^{0.5}K^{0.75}$. Capital, K, is fixed in the short run.
   a) Does this function exhibit the law of diminishing marginal returns? What type of returns to scale does it exhibit?
   b) The firm is presently using four units of capital. The unit prices of L and K are w = 2 and r = 25 respectively. Find the cost as a function of y. What output level minimises variable average and total costs?
   c) Without any calculations, what is the shape the short run average cost curve, increasing, decreasing, or U-shaped?
   d) Find the conditional factor demand functions. Find the cost curve when the input prices are as in part b) (as a function of y only).
   e) Draw the short- and long-run cost curves. The short-run with K = 4 (as in part b). For what output levels are the two costs the same, and for what levels is one higher than the other? What is the reason of that relationship?

5) Find the cost functions for the following technologies:
   a) $f(L, K) = \min\{2L, 3K\}$;
   b) $f(L, K) = 2L + 3K$;
   c) $f(L, K) = \max\{2L, 3K\}$;

6) Does the production cost depend on demand for the good? Explain.

7) A firm's technology is described by the production function $f(L, K) = L^aK^b$, and the prices of labour and capital are w and r.
   a) Find the long-run average cost function.
   b) Do long-run costs increase more or less than proportionally with output? Explain.

Answers

1.a) To minimise cost it must be $TRS = w/r$. $TRS = K/L = 7.5$; $w/r = 2.5$; so the firm is not operating efficiently. It should use more labour and less capital: TRS = 7.5 means using one more unit of labour allows the firm to spare 7.5 units of capital, keeping output unchanged; but one unit of labour costs only 2.5 units of capital, so the substitution would reduce costs.

1.b) The firm is currently producing $f(50, 375) = 136.93$. So
$L^{0.5}K^{0.5} = 136.93$ (1)
From a) we know efficient production requires:
$K/L = 2.5 \Leftrightarrow K = 2.5L$ (2)
Substituting in (1)
$L^{0.5} \times (2.5L)^{0.5} = 136.93 \Leftrightarrow L = 86.6$.
Substituting back in (2), $K = 216.51$

1.c) The initial cost is 5 x 50 + 2 x 375 = 1000. With the new input combination it is 5 x 86.6 + 2 x 216.51 = 866.03, so cutting costs by 13.4%.

1.d) This problem is formally identical to the standard consumer problem. Here the firm tries to reach the highest possible isoquant given its isocost line. We know the conditions that must be met are:
$5L + 2K = 1000$ (original isocost line – 1)
$TRS = K/L = w/r = 2.5$ (tangency condition – 2)
From (2), $K = 2.5L$ (3)
Substituting in (1)
$5L + 2 \times 2.5L = 1000 \Leftrightarrow L = 100$.
Substituting back in (2), $K = 250$.
The firm now produces $f(100, 250) = 158.1$ or 15% at the same cost than before.

2.a) The conditional factor demand functions give the input quantities that minimise the cost of achieving some output level. The cost-minimising input levels meet two conditions. The input combination must be on the required isoquant:
$f(L, K) = 4L^{0.25}K^{0.25} = y$ (1)
The isoquant must be tangent to the isocost line:
$TRS = w/r \Leftrightarrow K/L = w/r \Leftrightarrow K = wL/r$ (2)
Substituting (2) into (1):
$4L^{0.25} \times (wL/r)^{0.25} = y \Leftrightarrow L = (r/w)^{0.5}y^2/16$ (3)
Substituting (3) back in (2),
$K = (w/r)^{0.5}y^2/16$.

2.b) $C(w, r, y) = wL + rK = (rw)^{0.5}y^2/8$. It tells the minimum cost of producing y at input prices w and r.

2.c) It is just a scale parameter. Say, suppose we are measuring output in tons. Now we start measuring output in kilos (and keep the labour and capital units unchanged), then we would have to replace the 4 with 4000.

3.a) Proceeding as in the previous exercise, the cost-minimising conditions are

$f(L, K) = aL^{1/2}K^{1/6} = y$ (1)

$TRS = w/r \Leftrightarrow 3K/L = w/r \Leftrightarrow K = wL/(3r)$ (2)

Substituting (2) into (1):

$aL^{1/2} \times [wL/(3r)]^{1/6} = y \Leftrightarrow L^{2/3} = (3r/w)^{1/6}y/a \Leftrightarrow$

$\Leftrightarrow L = (3r/w)^{1/4}(y/a)^{3/2}$

Substituting (3) back in (2),

$K = (w/3r)^{3/4}(y/a)^{3/2}$.

3.b) $C(w, r, y) = wL + rK = (3^{1/4} + 3^{-3/4})w^{3/4}r^{1/4}(y/a)^{3/2}$.

3.c) $C(6, 2, y) = 8(y/a)^{3/2}$.

3.d) $C(6, 2, 1) = 8(1/a)^{3/2} = 80 \Leftrightarrow a = 0.1^{2/3} = 0.2154$.

4.a) Both inputs exhibit diminishing marginal product. Increasing returns to scale.

4.b) $L = y^2/8$, $C = 0.25y^2 + 100$. $y = 20$ minimises average cost. Average variable cost is increasing for all output levels.

4.c) U-shaped. There are fixed costs, and the average fixed cost is always decreasing. But there are diminishing marginal product, so average variable cost is always increasing. For small enough output level the effect of decreasing average fixed cost prevails, so average cost falls; for higher output levels the opposite is true, so average cost will rise.

4.d) $L = (r/1.5w)^{0.6}y^{0.8}$, $K = (1.5w/r)^{0.4}y^{0.8}$,
$C = (1.5^{-0.6} + 1.5^{0.4})w^{0.4}r^{0.6}y^{0.8}$. $C = 17.8427y^{0.8}$.

4.e) The two average costs are the same for $y = 16.33$; short-run average cost exceeds the long-run one for all other output levels (the two curves are tangent for $y = 16.33$). For $y = 16.33$ and at current input prices the optimal level of K is 4, the same as currently employed in the short run. For all other output levels, the optimal K is different from 4, so the short-run cost is higher than the long-run one.

5.a) Per unit of output the firm requires 0.5 units of labour and 1/3 units of capital. So the conditional factor demand functions are:

$L(w, r, y) = y/2$ and $K(w, r, y) = y/3$.

$C(w, r, y) = wL + rK = (w/2 + r/3)y$.

5.b) $TRS = 2/3$. So the firm will use labour only if $w/r < 2/3$, in which case $L = 0.5y$, and $C(w, r, y) = 0.5wy$. If $w/r > 2/3$ the firm uses capital only and $C(w, r, y) = ry/3$; if $w/r = 2/3$ the firm is indifferent between any combination of capital and labour, and $C(w, r, y) = ry/3 = 0.5wy$.

5.c) To produce y the firm will use either $K = y/3$ or $L = y/2$, whichever is cheaper. It turns out the cost function is exactly as in the previous case. (The only difference is that if $w/r = 2/3$ the firm will be indifferent between using labour only or capital only, but will not use a combination of both.)

6) Cost depends on the quantity produced and input prices only. So it does not directly depend on the demand for the good. The position of the demand curve will affect cost only if it influences input prices or the quantity produced by a firm. For instance, it is conceivable that increased demand for the good may in turn increase demand for inputs and so increase its prices. This would lead to higher costs.

7.a) The two condition for cost minimisation are

$f(L, K) = L^aK^b = y$ (1)

$TRS = w/r \Leftrightarrow aK/(bL) = w/r \Leftrightarrow K = wbL/(ra)$ (2)

Substituting (2) into (1):

$L^a \times (wbL/ra)^b = y \Leftrightarrow L = [(ra/wb)^by]^{1/(a+b)}$ (3)

Substituting (3) back in (2),

$K = = [(wb/ra)^ay]^{1/(a+b)}$

$C(w, r, y) = wL + rK = [\ [(a/b)^b + (b/a)^a](w^ar^by)\ ]^{1/(a+b)}$.

$AC(w, r, y) = [\ [(a/b)^b + (b/a)^a](w^ar^b)\ ]^{1/(a+b)}y^{(1-a-b)/(a+b)}$.

7.b) Average costs will increase as output increases, that is costs increases more than proportionally with output, if $1 - a - b > 0 \Leftrightarrow a + b < 1$. This makes sense: $a + b < 1$ means decreasing returns to scale, so average costs increase with output (see Varian). If $a + b > 1$ the opposite is true.