MATEMATICAL ECONOMICS - 2011/2012

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Contents

Warning

These notes are in a very preliminary form. Read these notes with some caution, as they likely to contain several mistakes and typos.

Corrections are greatly appreciated. Each section of the notes contains exercises. Some extra exercises can be found in the very last section.

1. Introduction

This part of the course concerns differential equations and difference equations. These equations are used to model dynamical processes, e.g., the evolutions of economical quantities changing in time. If the time is a continuous variable, then the process is modeled by an ordinary differential equation (ODE), whereas if the time is a discrete variable, then the process is modeled by a difference equation (DE). Sometimes, DE's are called *maps*.

Example (Compound interest). If an amount A is compounded annually at the market interest rate of r, then the payment after t years is given by

$$P_t = A(1+r)^t.$$

Here t is a discrete variable $t = 1, 2, \dots$. We see that P_t satisfies the following difference equation:

$$P_{t+1} = (1+r)P_t.$$

Now, if the same amount A is compound m times each year, then

$$P_t = A\left(1+\frac{r}{m}\right)^{mt}.$$

In the limit as m goes to infinity, we have

$$\lim_{m\to+\infty} P(t) = Ae^{rt}.$$

If we think of t as a continuous variable, then $P(t)$ is a solution of the differential equation

$$\frac{dP}{dt}(t) = rP(t).$$

The subject of the differential equations and the difference equation is extensive. In these lectures, we will be focusing on a part of the theory of these equations that is called 'qualitative analysis'. The aim is to obtain as much as possible information about a system without looking for explicit solutions.

2. Scalar ODE's

In this section, we are interested in ordinary differential equations. Here are some examples of ODE's:

- $dx/dt = -3x + 4 + e^{-t}$,
- $d^2x/dt + 4tdx/dt - 3(1-t^2) = 0$,
- $dx/dt + 3tx = e^x$.

2.1. **Main definitions.** In the following, the symbol x denotes a real-valued differentiable function $x \colon I \to \mathbb{R}$ on an open interval I of $\mathbb{R}$, whereas the symbol f denotes a real-valued continuous function $f \colon R \to \mathbb{R}$. We use the notation $\dot{x}(t)$ to denote dx/dt, the derivative of x at t. We are interested in equations of the form

$$\dot{x}(t) = f(x(t)) \qquad \text{for } t \in I, \tag{1}$$

where $t \mapsto x(t)$ is an unknown function. Equation (1) is called a *scalar autonomous differential equation.* The meaning of this terminology is as follows:

- 'scalar' means that x is 1-dimensional ($x \in \mathbb{R}$),
- 'autonomous' means that f does not depend explicitly on t,
- 'differential equation' is an equation involving the derivatives of x, the function x itself and other given functions.

A function x that satisfies relation (1) is called a *solution* of the differential equation (1).

Most of the time, we will be interested in solutions of (1) such that $x(t_0)$ equals a specific value $x_0 \in \mathbb{R}$ for a specific $t_0 \in \mathbb{R}$. The problem consisting in finding such a solution is called *initial value problem*,

$$\dot{x} = f(x), \qquad x(t_0) = x_0. \tag{2}$$

We can always take $t_0 = 0$. The reason is that if $x(t)$ is a solution of (2), then $y(t) = x(t - t_0)$ with $t_0 \in \mathbb{R}$ is the solution such that $y(t_0) = x_0$.

2.2. **Separation of variables.** To solve problem (2), we can argue as follows.

If $f(x_0) = 0$, then the function $x(t) = x_0$ for every $t \in \mathbb{R}$ is the wanted solution. Now, suppose that $f(x_0) \neq 0$. Since f is continuous, we have $f(x) \neq 0$ around x_0, and so as long as t is closed to t_0, we can divide both sides of (2) by $f(x(t))$. Hence,

$$\frac{\dot{x}(t)}{f(x(t))} = 1.$$

We then integrate both sides of the previous equation from t_0 to t,

$$\int_{t_0}^{t} \frac{\dot{x}(s)}{f(x(s))} ds = t - t_0,$$

and finally use the substitution $x = x(s)$ to compute the integral, obtaining

$$\int_{x_0}^{x(t)} \frac{dx}{f(x)} = t - t_0. \tag{3}$$

The left-hand side of this equation is a monotone function F evaluated at $x(t)$. If F^{-1} is the inverse of F, then we see that the solution $x(t)$ of (3) (for t close to t_0) is given by $x(t) = F^{-1}(t - t_0)$.

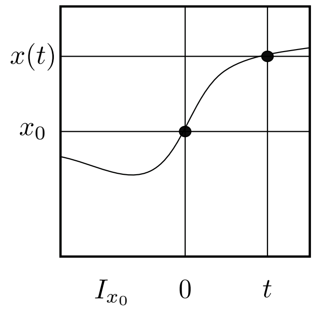


FIGURE 1. The solution $x(t, x_0)$ on the interval I_{x_0}

2.2.1. *Exercises.* Solve the following initial value problem using the method of separation of variables.

(1) $\dot{x} = -x, \quad x(0) = x_0$.
(2) $\dot{x} = x^2, \quad x(0) = x_0$.
(3) $\dot{x} = \sqrt{x}, \quad x(0) = x_0 \geq 0$. Is there just one solution for $x_0 = 0$?

2.3. **Existence and uniqueness of solutions.**

Definition 2.1. *The symbol C^0 denoted the sets of all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$, and the symbol C^1 denotes the subset of C^0 of all differentiable functions with continuous derivatives $f\colon \mathbb{R} \to \mathbb{R}$.*

As explained at the end of Subsection 2.1, there is no loss of generality in assuming that $t_0 = 0$. So, unless specified otherwise, $t_0 = 0$ from now on.

Theorem 2.2. *(1) Suppose that $f \in C^0$. Then for every x_0, there exist an interval (possibly infinite) $I_{x_0} = (a_{x_0}, b_{x_0})$ containing $t_0 = 0$ and a solution $x\colon I_{x_0} \to \mathbb{R}$ of the initial value problem (2).*
(2) Suppose that $f \in C^1$. Then in addition to (1), we have that the solution x is unique and differentiable with continuous derivative.

The largest possible interval I_{x_0} is called the *maximal interval of existence of the solution.*

We will often use the notation $x(t, x_0)$ to denote the solution of (2) with $x(0) = x_0$. In the next lemma, we summarize the main properties of $x(t, x_0)$.

Lemma 2.3. *The solution $x(t, x_0)$ has the following properties:*

(1) $x(t, x_0)$ is monotone in t,
(2) $x(t, x_0)$ is increasing in x_0, i.e, $x(t, x_0) < x(t, y_0)$ if $x_0 < y_0$,
(3) if $x(t, x_0)$ is bounded for every $t \geq 0$ ($t \leq 0$), then $b_{x_0} = +\infty$ ($a_{x_0} = -\infty$) and $\lim_{t\to+\infty} x(t, x_0) = \bar{x}$ ($\lim_{t\to-\infty} x(t, x_0) = \bar{x}$) with $\bar{x}$ being an equilibrium point (i.e., $f(\bar{x}) = 0$).

2.4. **Phase portrait.**

Definition 2.4. *Let $x_0 \in \mathbb{R}$, and let $x(t, x_0)$ be the solution with initial condition x_0. The set $\gamma(x_0) = \bigcup_{t \in (a_{x_0}, b_{x_0})} x(t, x_0)$ is called the* orbit of x_0. *The collection of the orbits of all points $x_0 \in \mathbb{R}$ is called the* phase portrait *of* (1).

Some special orbits:

- A point $\bar{x} \in \mathbb{R}$ is called an equilibrium point of (1) if $f(\bar{x}) = 0$. The constant function $x(t) = \bar{x}$ is a solution of (1).
- A solution $x(t, x_0)$ is called *periodic of period* T if $x(T, x_0) = x_0$. An equilibrium is a special case of a periodic orbit.

2.4.1. *Exercises.* Determine the phase portrait of the following differential equations:

(1) $\dot{x} = x$,
(2) $\dot{x} = x - x^3$,
(3) $\dot{x} = 1 + x$,
(4) $\dot{x} = x(1 - x)$,
(5) $\dot{x} = -x + x^3 + \lambda$ with $\lambda \in \mathbb{R}$,
(6) $\dot{x} = 1 - \sin x$.

2.5. **Equilibrium points and their stability.**

Definition 2.5. *An equilibrium point $\bar{x} \in \mathbb{R}$ of* (1) *is* stable *if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|x_0 - \bar{x}| < \delta$, then the solution $x(t, x_0)$ of* (1) *satisfies $|x(t, x_0) - \bar{x}| < \epsilon$ for every $t \geq 0$.*

Definition 2.6. *An equilibrium point $\bar{x} \in \mathbb{R}$ of* (1) *is* asymptotically stable *if it is stable and there exists $r > 0$ such that if $|x_0 - \bar{x}| < r$, then $\lim_{t \to +\infty} x(t, x_0) = \bar{x}$.*

Definition 2.7. *An equilibrium point $\bar{x} \in \mathbb{R}$ of* (1) *is called* unstable *if it is not stable.*

The following theorem is a stability criterion for equilibria in terms of the derivative of f.

Theorem 2.8. *Suppose that $f \in C^1$ and $\bar{x} \in \mathbb{R}$ is an equilibrium point of* (1).

(1) If $f'(\bar{x}) < 0$, then $\bar{x}$ is asymptotically stable.
(2) If $f'(\bar{x}) > 0$, then $\bar{x}$ is unstable.

An equilibrium point $\bar{x}$ is called *hyperbolic* if $f'(\bar{x}) \neq 0$, and non-hyperbolic if $f'(\bar{x}) = 0$.

Remark 2.9. *Note that Theorem 2.8 tells us about the stability of hyperbolic equilibrium points. There is no simple criterium for determining the stability of a non-hyperbolic equilibrium point.*

2.5.1. *Exercise.* Determine the type (hyperbolic or non-hyperbolic) and the stability of the equilibria in Exercises 2.4.1.

2.6. **Linear ODE's.** Consider the linear differential equation

$$\dot{x} = ax + b, \qquad a, b \in \mathbb{R}.$$

The equation is called *linear homogeneous* if $b = 0$, and *linear non-homogeneous* if $b \neq 0$.

To obtain the solution $x(t, x_0)$ (i.e, the solution satisfying the initial condition $x(0) = x_0$), one may argue as follows. If $a = 0$, then by integrating, we immediately obtain

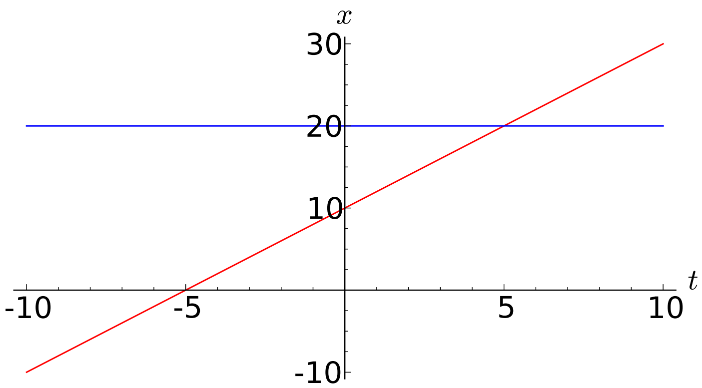
$$x(t, x_0) = x_0 + bt.$$

If $a \neq 0$, then first find the equilibrium $\bar{x} = -b/a$. Then define $y(t) = x(t) - \bar{x}$. Thus $\dot{y} = \dot{x} = ax + b = a(x - \bar{x}) = ay$. Now, the solution of $\dot{y} = ay$ such that $y(0) = y_0$ is given by $y(t, y_0) = y_0 e^{at}$, and note that $y_0 = x_0 - \bar{x}$. Finally, $x(t, x_0) = y(t, y_0) + \bar{x} = (x_0 - \bar{x})e^{at} + \bar{x}$, i.e.,

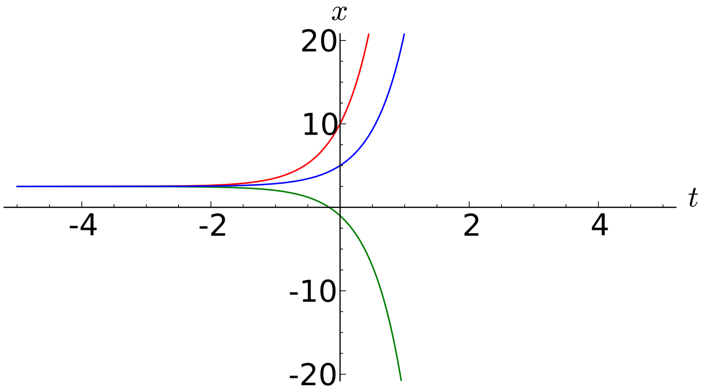
$$x(t, x_0) = \left(x_0 + \frac{b}{a}\right) e^{at} - \frac{b}{a}.$$

Examples.

(1) Suppose that $a = 0$. The next figure depicts the solutions $x(t, 10)$ for $b = 2$ (red) and the solution $x(t, 20)$ for $b = 0$ (blue).



(2) Suppose that $a = 2$ and $b = 1$. Then $\bar{x} = -1/2$ is the (unique) equilibrium point, and the solutions $x(t, 1)$, $x(t, -1)$ and $x(t, -1/2)$ are given by



2.6.1. *Exercises.*

(1) We assume that p, s, d are C^1 functions from $\mathbb{R}$ to $\mathbb{R}$, and that they represent the following quantities
- $p(t)$ = price of some good at time t,
- $d(t)$ = demand at time t,
- $s(t)$ = supply at time t.

These quantities are related by the following equations:

$$\dot{p}(t) = \alpha\left(d(t) - s(t)\right), \qquad \alpha > 0, \tag{4}$$

$$d(t) = A + Bp(t), \qquad A > 0, B < 0, \tag{5}$$

$$s(t) = C + Dp(t), \qquad C < 0, D > 0. \tag{6}$$

Substituting (5) and (6) in (7), we obtain the linear non-homogeneous equation

$$\dot{p} = \alpha(B - D)p + \alpha(A - C). \tag{7}$$

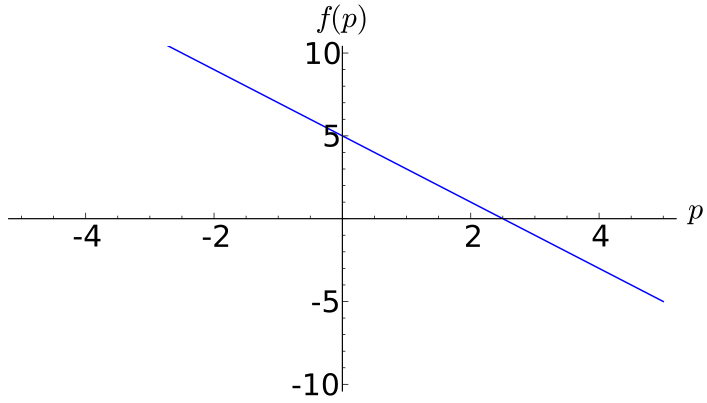
Let $a = \alpha(B - D) < 0$ and $b = \alpha(A - C) > 0$. The unique equilibrium point of this equation is $\bar{p} = -(A - C)/(B - D)$. From Subsection 2.6, the solution $p(t, p_0)$ with initial condition p_0 is then given by

$$p(t, p_0) = (p_0 - \bar{p})\, e^{at} - \bar{p}.$$

Since $a < 0$, we see that the price $p(t)$ converges to the equilibrium point $\bar{p}$ independently on he initial value p_0:

$$\lim_{t \to +\infty} p(t) = -\frac{A - C}{B - C}.$$

The same conclusion can be obtained by the Phase Portrait Analysis. The figure below depicts the graph of $\alpha(B - D)p + \alpha(A - C)$ for $\alpha(B - D) = -2$ and $\alpha(A - C) = 5$. The equilibrium point $\bar{p} = 2.5$ is asymptotically stable.



(2) Compound interest. If an amount A is compounded annually at the market interest rate r, then the payment after t years is given by

$$P_t = A(1+r)^t.$$

If the same amount is compound m times each year, then

$$P_t = A\left(1+\frac{t}{m}\right)^{mn}.$$

In the limit as m goes to infinity, we obtain $P(t) = Ae^{rt}$, which can be seen as the solution of the differential equation

$$\dot{P}(t) = rP(t).$$

Now, suppose that in addition to the interest rate rP received, there is a constant rate of deposit d. Write the new differential equation for P, and find the solution $P(t, P_0)$, i.e., the solution through P_0 at time $t_0 = 0$. Determine the phase portrait of the equation and the stability of its equilibrium point.

(3) A simple continuous price-adjustment demand and supply model is given by

$$\begin{aligned}
\dot{p}(t) &= \alpha(d(t) - s(t)), & \alpha > 0,\\
d(t) &= A + Bp(t), & B < 0,\\
s(t) &= C + Dp(t), & D > 0,
\end{aligned}$$

where $p(t), d(t)$ and $s(t)$ denotes the price, demand and supply at time t, respectively. Find and solve the differential equation for $p(t)$. Determine the phase portrait of the differential equation and the stability of its equilibrium point. Solve the same exercise when the demand $d(t)$ depends also on the variation of $p(t)$, i.e., when $d(t) = a + Bp(t) + F\dot{p}$ with $F \neq 0$.

(4) Assume that a population $p(t)$ grows at a constant rate k. This means that $p(t)$ satisfies the following differential equation:

$$\dot{p}(t) = kp(t).$$

Find the solution $p(t, p_0)$, determine the phase portrait of the equation and the stability of its equilibrium point.

(5) According to a continuos version of the Harrod-Domar economy growth model, the relation between the savings S, the income Y and the investment I is given by

$$S = sY, \quad I = \nu \dot{Y}, \quad I = S,$$

where s and ν are constants denoting the average propensity to save and the coefficient of the investment relationship, respectively. Derive and solve the differential equation for $Y(t)$. Determine its phase portrait and the stability of its equilibrium point.

3. Scalar DE's

Difference equations (DE's) are the analog of differential equations when the time is a discrete variable $n = 0, 1, \ldots$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a C^1 real-valued function.

3.0.2. *Example.* The following is an example of a difference equation arising from a financial problem.

Let p_n be the price of some financial assets at time $n = 0, 1, 2, \ldots$. Suppose that the variation of p_n in time is given by the following *arbitrage condition*:

$$(1 + r)p_n = d + p^e_{n+1}, \tag{8}$$

where $r > 0$ is the rate of return, $d > 0$ is the dividend, and p^e_{n+1} is the expected price at time $n + 1$. Suppose also that the agents have perfect foresight, i.e., they know that the mechanism of price formation is given by the following relation

$$p^e_{n+1} = p_{n+1}. \tag{9}$$

We want to determine how p_n varies in time.

By combining (8) and (9), we obtain a difference equation for p_n only:

$$(1 + r)p_n = d + p_{n+1}.$$

This equation can be written as

$$p_{n+1} = F(p_n), \qquad \text{where } F(p) = (1 + r)p + d.$$

This is the DE describing the evolution of p_n.

3.1. **General form.** We are interested in DE's of the form

$$x_{n+1} = F(x_n), \tag{10}$$

where $F : \mathbb{R} \to \mathbb{R}$ is a continuous function, called *map*.

The solution of Equation (10) with the initial condition x_0 is obtained recursively:

$$x_0, F(x_0), F^2(x_0), \ldots,$$

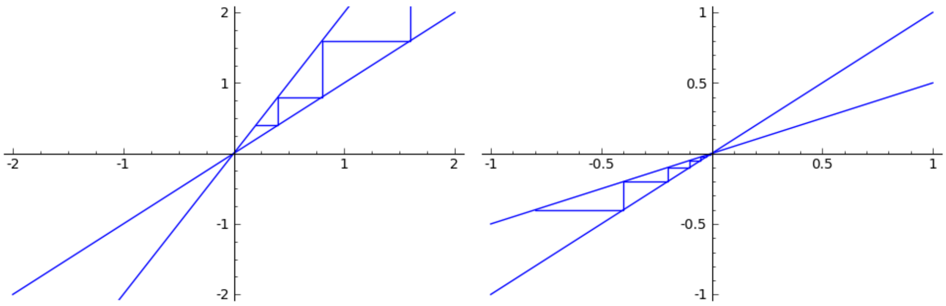


FIGURE 2. (A) $x_{n+1} = 2x_n$ with $x_0 = 0.2$. (B) $x_{n+1} = x_n/2$ with $x_0 = -0.8$.

where $F^n(x_0) = \underbrace{F \circ F \circ \cdots \circ F}_{n \text{ times}}$.

Definition 3.1. *The union of all elements $x_0, F(x_0), F^2(x_0), \ldots$ is called the* positive orbit of x_0*, and is denoted by $\gamma^+(x_0)$.*

Definition 3.2. *A point $\bar{x} \in \mathbb{R}$ is called a* fixed point of the map F *if $F(\bar{x}) = \bar{x}$.*

Remark 3.3. *Note that $\bar{x}$ is a fixed point of F if and only if $\gamma^+(\bar{x}) = \bar{x}$.*

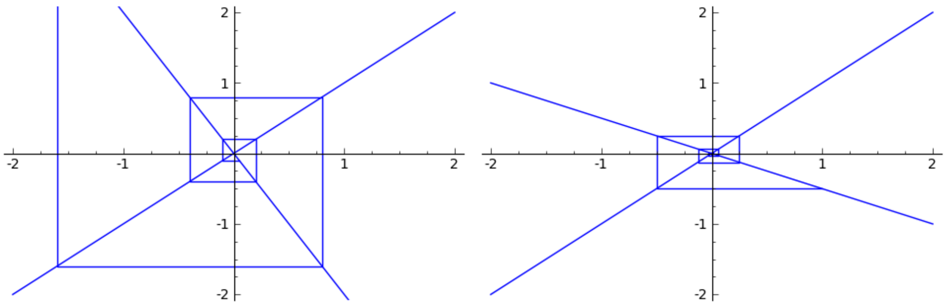
3.2. **Stair-step diagram.** The stair-step diagram is a geometrical method for depicting the orbits of a DE. The method is illustrated in the following examples.

3.2.1. *Examples.*

(1) $x_{n+1} = 2x_n$. The orbits of this map can be computed explicitly. By iterating F, we obtain $x_n = 2^n x_0$ for $x_0 \in \mathbb{R}$. The step-stair diagram for this map is depicted in Fig. 2(A).
(2) $x_{n+1} = x_n/2$. The orbits of this maps are $x_n = 2^{-n} x_0$ for $x_0 \in \mathbb{R}$ (see Fig. 2(B))
(3) $x_{n+1} = -2x_n$. The orbits of this maps are $x_n = (-2)^n x_0$ for $x_0 \in \mathbb{R}$ (see Fig. 3(A)). Compare these orbits with those of the previous examples. Note the oscillatory behavior of the orbits in this example and the next.
(4) $x_{n+1} = -x_n/2$. The orbits of this maps are $x_n = (-2)^{-n} x_0$ for $x_0 \in \mathbb{R}$ (see Fig. 3(B)).

3.3. **Stability.** As for equilibrium points of differential equations, we can define the notions of a stability, instability and asymptotic stability for fixed points.

Definition 3.4. *Let $\bar{x} \in \mathbb{R}$ be a fixed of the map F. Then $\bar{x}$ is called*



(A) (B)

FIGURE 3. (A) $x_{n+1} = -2x_n$ with $x_0 = 0.01$. (B) $x_{n+1} = -x_n/2$ with $x_0 = 1$.

(1) stable *if for every* $\epsilon > 0$*, there exists* $\delta > 0$ *such that if* $|x_0 - \bar{x}| < \delta$*, then the orbit* x_n *satisfies* $|x_n - \bar{x}| < \epsilon$ *for every* $n \geq 0$*;*
(2) asymptotically stable *if it is stable and there exists* $r > 0$ *such that if* $|x_0 - \bar{x}| < r$*, then the orbit* x_n *satisfies* $\lim_{t\to+\infty} x_n = \bar{x}$*;*
(3) if it is not stable.

The following theorem is a stability criterion in terms of the derivative of F.

Theorem 3.5. *Suppose that* $F \in C^1$ *and* $\bar{x} \in \mathbb{R}$ *is a fixed point of* F*.*
(1) If $|F'(\bar{x})| < 1$*, then* $\bar{x}$ *is asymptotically stable.*
(2) If $|F'(\bar{x})| > 1$*, then* $\bar{x}$ *is unstable.*

A fixed point $\bar{x}$ is called *hyperbolic* if $|F'(\bar{x})| \neq 1$, and *non-hyperbolic* if $|F'(\bar{x})| = 1$.

Remark 3.6. *There is no simple criterium for determining the stability of a non-hyperbolic fixed points.*

3.3.1. *Exercises.* Find the fixed points of the map F and determine their stability. Some of the fixed points are non-hyperbolic, and therefore Theorem 3.5 cannot be used. Use instead the stair-step diagram.
(1) $F(x) = x + x^2$.
(2) $F(x) = -x + 3x^2$.

3.4. **Linear maps.** A linear difference equation is an equation of the form:

$$x_{n+1} = ax_n + b, \qquad a, b \in \mathbb{R}. \tag{11}$$

The equation is called *homogeneous* if $b = 0$, and *non-homogeneous* if $b \neq 0$.

Solutions of these equations can be computed explicitly. Note first that Equation (11) has a (unique) fixed point $\bar{x} = b/(1-a)$ if and only if $a \neq 1$. The orbit of (11) is given by

$$x_n = \begin{cases} x_0 + nb & \text{if } a = 1, \\ a^n(x_0 - \bar{x}) + \bar{x} & \text{otherwise.} \end{cases} \tag{12}$$

This includes the case $a = 0$ for which the orbit consists of the fixed point $\bar{x} = b$.

3.4.1. *Exercise.* Derive Formula (12). This can be done using a method similar to that one used to obtain the solutions of linear differential equations in Subsection 2.6.

4. Planar ODE's

Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a function. Also, let I be an interval of $\mathbb{R}$, and let $x\colon I \to \mathbb{R}^2$ be a differentiable function. We are interested in the solutions of the autonomous (f does not depend explicitly on t) differential equation

$$\dot{x} = f(x), \qquad t \in I. \tag{13}$$

4.1. **Linear ODE's.** More specifically, we are interested in the case $f(x) = Ax$ with A being a 2×2 matrix with real coefficients and x being a vector of $\mathbb{R}^2$:

$$\dot{x} = Ax. \tag{14}$$

If we write

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}, \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}. \end{pmatrix}$$

then Equation (14) takes the form

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2, \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2. \end{aligned}$$

4.2. **General properties of linear systems.**

1. Existence and uniqueness: the solution $x(t, x_0)$ of Equation (14) with initial condition $x(0) = x_0 \in \mathbb{R}^2$ exists and its unique. Moreover, its maximal interval of existence is the entire real line $\mathbb{R}$.
2. Superposition Principle: if x and y are two solutions of (14), then every linear combination $c_1x + c_2y$ with $c_1, c_2 \in \mathbb{R}$ is a solution as well. This is simple to prove. Let $z = c_1x + c_2y$. Then $\dot{z} = c_1\dot{x} + c_2\dot{y}$. Since x and y are solutions of (14), we have $\dot{z} = c_1Ax + c_2Ay$. But $c_1Ax + c_2Ay = A(c_1x + c_2y) = Az$, and we can conclude that $\dot{z} = Az$, i.e., z is a solution.

(3) In analogy to the scalar case, the solution of (14) with initial condition $x_0 \in \mathbb{R}^2$ is given by

$$x(t, x_0) = e^{tA}x_0, \quad t \in \mathbb{R},$$

where e^{tA} is a matrix, which is defined in the next subsection.

4.3. **Exponential of a matrix.** It is a fact that the series $\sum_{n=0}^{+\infty} A^n/n!$ converges for every 2×2 matrix A. This allows us to define the exponential of a matrix as follows.

Definition 4.1. *We define*

$$e^A = \sum_{n=0}^{+\infty} \frac{1}{n!} A^n.$$

Of course if A is a matrix and t is a real number, then tA is still a matrix. The main properties of the matrix e^{tA} are the following:

(1) $e^{(s+t)A} = e^{sA}e^{tA}$ for $s, t \in \mathbb{R}$,
(2) $de^{tA}/dt = Ae^{tA} = e^{tA}A$,
(3) if $AB = BA$ (i.e., A and B commute), then $e^{t(A+B)} = e^{tA}e^{tB}$.

4.3.1. *Exercise.* Show that if $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 & 0 \\ -1 & 0 \end{smallmatrix}\right)$, then $e^{t(A+B)} \neq e^{tA}e^{tB}$.

4.4. **Exponential of Normal Jordan Forms.**

Definition 4.2. *Every matrix having one of the following three forms is called a* Jordan Normal Form*,*

$$(i) \quad \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad (ii) \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \qquad (iii) \quad \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

where $\lambda_1, \lambda_2, \lambda, \alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$.

We now compute e^{tA} when A is a Normal Jordan Form.

Form (i): It follows directly from the definition of e^{tA} that

$$e^{tA} = \begin{pmatrix} \sum_{n=0}^{+\infty} \dfrac{(t\lambda_1)^n}{n!} & 0 \\ 0 & \sum_{n=0}^{+\infty} \dfrac{(t\lambda_2)^n}{n!} \end{pmatrix} = \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix}.$$

Form (ii): We can write $A = I + \lambda N$, where $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $N = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Since I and N commute, it follows from Property (3) of e^{tA} that

$$e^{tA} = e^{\lambda I}e^{tN} = e^{\lambda t}e^{tN}.$$

Now, we see that $N^2 = 0$ (i.e., N^2 is the matrix with zero entries). This implies that $N^k = 0$ for $k \geq 2$, and so

$$e^{tN} = I + tN = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Hence,

$$e^{tA} = e^{t\lambda}\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix} = \begin{pmatrix}e^{t\lambda} & te^{t\lambda}\\ 0 & e^{t\lambda}\end{pmatrix}.$$

Form (iii): We can write $A = \alpha I + \beta K$, where $K = \left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$. Since I and K commute, Property (3) of e^{tA} implies that

$$e^{tA} = e^{\alpha t}e^{\beta t K}.$$

Now, check that $K^2 = -I$ and $K^3 = -K$. From this, we get $K^{2n} = (-1)^n I$ and $K^{2n+1} = (-1)^n K$, and so

$$\begin{aligned}
e^{tK} &= \sum_{n=0}^{\infty}\frac{1}{(2n)!}(\beta t)^{2n}K^{2n} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}(\beta t)^{2n+1}K^{2n+1}\\
&= \left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(\beta t)^{2n}\right) I + \left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(\beta t)^{2n+1}\right) K\\
&= \cos(\beta t) I + \sin(\beta t) K = \begin{pmatrix}\cos(\beta t) & \sin(\beta t)\\ -\sin(\beta t) & \cos(\beta t)\end{pmatrix}.
\end{aligned}$$

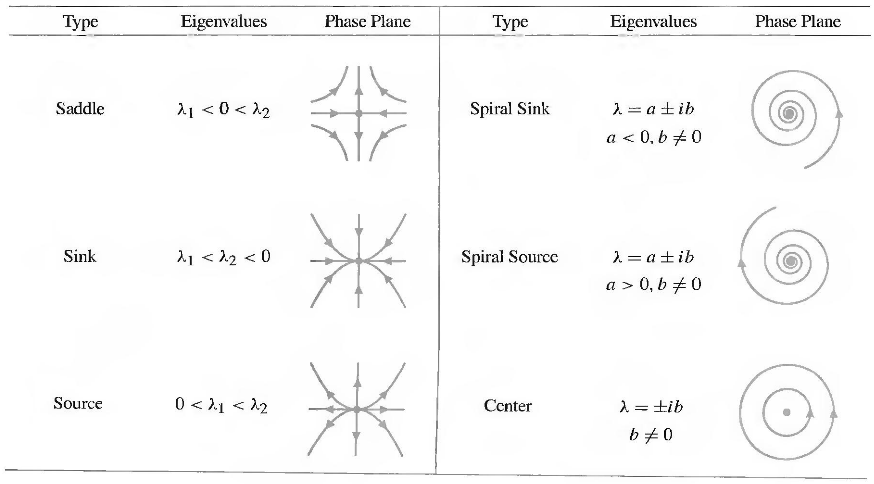
Finally,

$$e^{tA} = e^{\alpha t}\begin{pmatrix}\cos(\beta t) & \sin(\beta t)\\ -\sin(\beta t) & \cos(\beta t)\end{pmatrix}.$$

4.5. **Phase portrait.** We now draw the phase portrait of the differential equation $\dot{x} = Ax$ when A is one of the Normal Jordan Forms introduced in Subsection 4.4. Although the phase portrait is the collection of all the orbits of the equation, we do not need to plot all the of them, but only a few representative ones. Since we know that the general solution of the equation is $x(t, x_0) = e^{tA}x_0$ with $x(0) = x_0$, all that we need to do is to understand the geometry of the transformation of the plane $x_0 \mapsto e^{tA}x_0$, sending the vector x_0 into the new vector $e^{tA}x_0$.

Form (i): It is quite easy to understand the geometrical effect of the transformation e^{tA} in this case. Its effect is that of multiplying the first component of the vector x by $e^{t\lambda_1}$ and the second component of x by $e^{t\lambda_2}$. Depending on the sign of λ_1 and λ_2, the phase portrait is depicted in Fig. 4 (cases: saddle, sink and source).

Form (ii): The transformation e^{tA} can be thought as the compositions of two transformations: $e^{t\lambda}x$ and $\left(\begin{smallmatrix}1 & t\\ 0 & 1\end{smallmatrix}\right)x$. The first transformation expands or contracts x depending on the sign of λ, whereas the second transformation 'slides' the vector $x = (x_1, x_2)$ along the horizontal line $y = x_2$. The overall effect of e^{tA} produces the phase portrait (improper node) depicted in Fig. 5.



Type	Eigenvalues	Phase Plane	Type	Eigenvalues	Phase Plane
Saddle	$\lambda_1 < 0 < \lambda_2$		Spiral Sink	$\lambda = a \pm ib$ $a < 0, b \neq 0$	
Sink	$\lambda_1 < \lambda_2 < 0$		Spiral Source	$\lambda = a \pm ib$ $a > 0, b \neq 0$	
Source	$0 < \lambda_1 < \lambda_2$		Center	$\lambda = \pm ib$ $b \neq 0$	

FIGURE 4. Phase Portraits.

Form (iii): The geometry of $e^{tA}x$ is the combination of the expansion or contraction generated by $e^{\alpha t}$ with the rotation of the plane generated by the matrix $\begin{pmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{pmatrix}$ (clockwise if $\beta > 0$ and counterclockwise if $\beta < 0$). The phase portrait is depicted in Fig. 4 (cases: spiral sink, spiral source and center).

4.6. **Change of coordinates.** Suppose that x is a solution of the differential equation $\dot{x} = Ax$. Let P be an invertible real 2×2 matrix, and define $y = P^{-1}x$. Then, y is a solution of the differential equation:

$$\dot{y} = P^{-1}\dot{x} = P^{-1}Ax = P^{-1}APy.$$

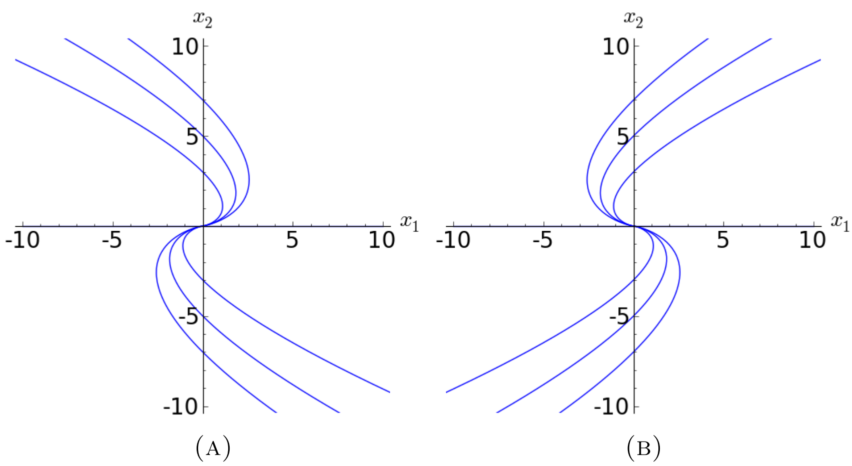


FIGURE 5. Improper Node. (A) $\lambda < 0$. (B) $\lambda > 0$.

The general solution of this equation is $y(t) = e^{tP^{-1}AP}y_0$ for $y_0 \in \mathbb{R}^2$. This implies that $x(t) = Pe^{tP^{-1}AP}P^{-1}x_0$, where $x_0 = Py_0 = x(0)$. But we know that solution $x(t, x_0)$ is given by $x(t, x_0) = e^{tA}x_0$, and so we conclude that

$$e^{tA} = Pe^{tP^{-1}AP}P^{-1}.$$

Now, suppose that given a real matrix A, we can find an invertible matrix P such that $P^{-1}AP$ is a Jordan Normal form. So in order to compute e^{tA}, we do not have to compute directly e^{tA}, but we can simply compute $Pe^{tP^{-1}AP}P^{-1}$, and we know that from Subsection 4.4 that $e^{tP^{-1}AP}$ is one of the matrices:

$$\begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix}, \qquad e^{t\lambda}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad e^{\alpha t}\begin{pmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{pmatrix}.$$

4.6.1. *Exercise.* Consider the linear differential equation

$$\dot{x} = \begin{pmatrix} 5 & -4 \\ 4 & 5 \end{pmatrix},$$

and the change of coordinates $y = Px$ with $P = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. Find the differential equation in the new coordinates y, compute the general solution for this equation, and finally derive the general solution in the coordinates x.

4.7. **Jordan Decomposition Theorem.**

Theorem 4.3. *Suppose that A is a real 2×2 matrix. There exists and invertible real 2×2 matrix P such that $P^{-1}AP = J$, and J is one of the following matrices:*

$$(i) \quad \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad (ii) \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \qquad (iii) \quad \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

with $\lambda_1, \lambda_2, \lambda, \alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$. The matrix J is called a Normal Jordan form.

We now explain how to compute the matrix P. The procedure consists of three steps:

Step 1: Find the eigenvalues of A, which are solutions of the characteristic equation:

$$\det(A - \lambda I) = \lambda^2 - \mathrm{tr}(A)\lambda - \det(A) = 0, \tag{15}$$

where $\mathrm{tr}(A)$ and $\det(A)$ are the trace and determinant of A, respectively. This is a quadratic equation with real coefficients, and so it has two solutions λ_1 and λ_2 that can be of one of the following types:

- **(a):** λ_1, λ_2 real and $\lambda_1 \neq \lambda_2$,
- **(b):** $\lambda_1 = \lambda_2 = \lambda$ real,
- **(c):** $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$ with α, β real and $\beta \neq 0$, i.e., λ_1 and λ_2 are complex conjugate.

Step 2: Find the eigenvectors of A. This can be done for each case (a), (b) and (c) as follows.

(a): Since $\lambda_1 \neq \lambda_2$, the matrix A is diagonalizable. This means that A has two linearly independent eigenvectors v_1 and v_2 corresponding to the eigenvalues λ_1 and λ_2, respectively. These vectors are non-zero solutions of the equations:

$$(A - \lambda_i I)\, v_i = 0, \qquad i = 1, 2.$$

(b): We have two subcases. The first corresponds to the situation when A admits two linearly independent eigenvectors v_1 and v_2, that is, when two linearly independent vectors v_1 and v_2 are solutions of the equation

$$(A - \lambda I)\, v = 0. \tag{16}$$

The second subcase corresponds to the situation when any two of non-zero solutions of equation (16) are linearly dependent. In this case, let v_1 be a non-zero solution of (16), and let v_2 be any non-zero vector such that

$$(A - \lambda I)\, v_2 = v_1. \tag{17}$$

The vector v_1 is an eigenvector of A, and v_2 is called a *generalized eigenvector of* A.

(c): Let v be an eigenvector of A corresponding to the eigenvalue $\alpha + i\beta$. It turns out that the components of v are complex numbers. So we can write $v = v_1 + iv_2$, where v_1 and v_2 are vectors with real components.

Step 3: Let v_1 and v_2 be the vectors computed for each case in Step 2. Then $P = (v_1|v_2)$. This means that v_1 and v_2 are the first column and the second column of P, respectively. From the construction of v_1 and v_2 in Step 2, these vectors are linearly independent (can you explain why?), and so P is invertible. Finally, the Jordan Normal form J associated to A is given by $J = P^{-1}AP$.

4.8. **Stability.**

Theorem 4.4. *Let A be a real 2×2 matrix. Then the origin $(0, 0)$ is always an equilibrium point of the equation $\dot{x} = Ax$. Furthermore,*

(1) if all the eigenvalues of A have negative real parts, then the origin is asymptotically stable;
(2) if at least one of the eigenvalues of A has positive real part, then the origin is unstable.

4.8.1. *Exercises.* Consider the linear differential equation $\dot{x} = Ax$. For each of the cases below, find the matrix P and the Jordan Normal form J for A. Then sketch the phase portrait of the equation in the new coordinates $y = P^{-1}x$, and determine the stability of the equilibrium point $(0,0)$. Finally, compute $e^{tA} = Pe^{tJ}P^{-1}$. How many equilibrium points does the equation have in exercise iv)?

$$i)\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad ii)\ \frac{1}{2}\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}, \qquad iii)\ \begin{pmatrix} 0 & -2 \\ 8 & 0 \end{pmatrix}, \qquad iv)\ \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

4.9. **Non-homogeneous linear differential equations.** Let A be an invertible real 2×2 matrix, and let b a vector of $\mathbb{R}^2$. Consider the linear differential equation

$$\dot{x} = Ax + b. \tag{18}$$

Since A is invertible, this equation has a unique equilibrium $\bar{x} = -A^{-1}b$. The solution with initial condition $x(0) = x_0$ is given by

$$x(t, x_0) = \bar{x} + e^{tA}(x_0 - \bar{x}), \qquad t \in \mathbb{R}. \tag{19}$$

4.9.1. *Exercise.* Check that (19) is the solution of Equation (18) with initial value $x(0) = x_0$.

5. Nonlinear systems

Suppose that $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 function. That is, we assume that there are two C^1 functions f_1 and f_2 from $\mathbb{R}^2$ to $\mathbb{R}$ such that $f = \left(\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right)$. The differential of f at $x = (x_1, x_2) \in \mathbb{R}^2$ is the matrix

$$Df(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}.$$

We are interested in the autonomous differential equation

$$\dot{x} = f(x). \tag{20}$$

Definition 5.1. *A point $\bar{x} \in \mathbb{R}^2$ is called an* equilibrium *of* (20) *if $f(\bar{x}) = 0$. An equilibrium point $\bar{x}$ is called* hyperbolic *if all the eigenvalues of $Df(\bar{x})$ have non-zero real part.*

Theorem 5.2. *The stability criterion for equilibria of linear systems (Theorem* (4.4)*) applies to every hyperbolic equilibrium $\bar{x}$ of* (20) *with the matrix A replaced by $Df(\bar{x})$.*

The previous theorem is a corollary of a more general result that roughly says that the phase portrait of Equation (20) in a neighborhood of a hyperbolic equilibrium $\bar{x}$ 'looks' like the phase portrait of the linear differential equation

$$\dot{x} = Df(\bar{x})x. \tag{21}$$

This equation is called the *variational linear variational equation at $\bar{x}$.*

5.0.2. *Exercises.* Find the equilibria and determine their stability for the following planar differential equations:

(1) $\dot{x}_1 = 1 - x_1x_2, \qquad \dot{x}_2 = x_1 - x_2^2,$
(2) $\dot{x}_1 = 2x_1 - x_1^2 - x_1x_2, \qquad \dot{x}_2 = -x_2 + x_1x_2,$
(3) $\dot{x}_1 = \sin(x_1 + x_2), \qquad \dot{x}_2 = e^{x_1} - 1,$
(4) $\dot{x}_1 = x_1 - x_1^3 - x_1x_2, \qquad \dot{x}_2 = 2x_2 - x_2^5 - x_2x_1^4.$

6. Extra exercises

6.1. **Scalar ODE's.**

(1) Solve the initial value problem using the method of separation of variables.

$$\begin{aligned} (i) \quad & \dot{x} = \frac{1}{x^2}, & x(0) \neq 0, \\ (ii) \quad & \dot{x} = x(x-2), & x(0) = x_0, \\ (iii) \quad & \dot{x} = \frac{1}{2\sqrt{x}}, & x(0) = x_0 \geq 0. \end{aligned}$$

(2) For each of the following differential equations find all the equilibrium points and determine whether they are stable, asymptotically stable or unstable. Also, draw the phase portrait.

$$\begin{aligned} (i) \quad & \dot{x} = x^3 - 3x, \\ (ii) \quad & \dot{x} = x^4 - x^2, \\ (iii) \quad & \dot{x} = \cos x, \\ (iv) \quad & \dot{x} = \sin^2 x, \\ (v) \quad & \dot{x} = |1 - x^2|. \end{aligned}$$

(3) The following differential equations depends on a parameter a. Plot the phase portrait for $a = -1$, $a = 0$ and $a = 1$.

$$\begin{aligned} (i) \quad & \dot{x} = x^2 - ax, \\ (ii) \quad & \dot{x} = x^3 - ax. \end{aligned}$$

(4) Solve the following linear non-homogeneous equations

$$\begin{aligned} (i) \quad & \dot{x} = 2x + 3, & x(0) = 10, \\ (ii) \quad & \dot{x} = -x + 2, & x(0) = -10, \\ (iii) \quad & \dot{x} = 3x + 10, & x(0) = 2. \end{aligned}$$

6.2. **Scalar maps.**

(1) For each of the following difference equations, draw the stair-step diagram and plot some iterations. Establish whether the fixed point is stable, asymptotically stable or unstable. Explain why. In which of these examples does the system oscillate

around the fixed point?

$$(i) \quad 10 - 3x_n = 2 + x_{n-1},$$
$$(ii) \quad 25 - x_{n+1} = 3 + 4x_{n-1},$$
$$(iii) \quad 45 - 2.5x_{n+1} = 5 + 7.5x_{n-1}.$$

(2) For the following difference equations, draw the stair-step diagram, and iterates 4 times the initial condition $x_0 = .4$. Determine whether the fixed points are stable, asymptotically stable or unstable.

$$(i) \quad x_{n+1} = 4x_n(1 - x_n),$$
$$(ii) \quad x_{n+1} = x_n^2 - 2,$$
$$(iii) \quad x_{n+1} = -2\left|x - \frac{1}{2}\right| + 1.$$

6.3. **Planar ODE's.**

(1) Sketch the phase portrait of the equation $\dot{x} = Ax$ for the following matrices. Determine the stability of the origin, and compute the exponential matrix e^{tA}.

$$a) \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad b) \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}, \qquad c) \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix},$$
$$d) \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix}, \qquad e) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad f) \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

(2) For each of the following linear equations $\dot{x} = Ax$
   (a) Find the eigenvalues and eigenvectors of A.
   (b) Find the matrix P such that $J = P^{-1}AP$ is a Jordan Normal form.
   (c) Compute the exponential matrices e^{tJ} and e^{tA}.
   (d) Find the solution $x(t, x_0)$ with initial condition x_0.
   (e) Sketch the phase portrait for the system $\dot{y} = Jy$.
   (f) Determine the stability of the origin $(0, 0)$.

$$a) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad b) \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad c) \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix},$$
$$d) \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}, \qquad e) \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix}, \qquad f) \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

(3) Solve the initial value problem: $\dot{x}_1 = -4x_2$, $\dot{x}_2 = x_1$ with $x_1(0) = 0$ and $x_2(0) = -7$.

(4) Find all the solutions of the linear non-homogeneous system: $\dot{x}_1 = x_2$, $\dot{x}_2 = 2 - x_1$.

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