Mathematical Economics

Deterministic dynamic optimization

Continuous time

Paulo Brito

pbrito@iseg.utl.pt

3.12.2012

Contents

1 Introduction

In these notes the state of a system is represented by function $x : \mathbb{T} \rightarrow \mathbb{R}$, where the domain of the independent variable is a subset of $\mathbb{R}_+$, $\mathbb{T} \subseteq \mathbb{R}_+$, and $x(t) \in \mathbb{R}$, that is, we deal with scalar problems. We represent the path or orbit of x by $x(.) = (x(t))_{t\in\mathbb{T}}$ to distinguish it from the value taken at time t, $x(t)$.

Again, we want to find optimal paths, $x^*(.) = (x^*(t))_{t\in\mathbb{T}}$ which solve problems of dynamic optimization. Three types of problems are considered

1. calculus of variations
2. optimal control by using the Pontriyagin's principle
3. dynamic programming.

2 Calculus of variations

Calculus of variations problems have three components:

1. the value functional
$$V[x(.)] = \int_0^T F(t, x(t), \dot{x}(t))dt \tag{1}$$
where we assume throughout that $F(.)$ is continuous, differentiable and smooth ($F \in C^2(\mathbb{R}^3)$) in all its arguments and is concave in $(x, \dot{x})$;

2. data related to the initial state, where $x(0) = x_0$ is given

3. some information, or data, on the terminal time, T, and/or state , $x(T)$.

We will consider the following problems:

- the simplest problem: find $\max_x V[x]$ such that T is finite and $x(0) = x_0$ and $x(T) = x_T$ are given;

- the simplest free terminal state problem: find $\max_x V[x]$ such that T is finite and given and $x(0) = x_0$ is given and $x(T)$ is free;

- infinite horizon discounted problem: $F(x(t), \dot{x}(t), t) = e^{-\rho t} f(x(t), \dot{x}(t))$, where $\rho > 0$ and $T = \infty$

2.1 The simplest problem

The problem: find a path $x^*(.) = (x^*(t))_{t \in \mathbb{T}}$ such that

$$\begin{cases} \max_{x(.)} V[x(.)] \\ x(0) = x_0 \text{ given} \quad t = 0 \\ x(T) = x_T \text{ given} \quad t = T \end{cases} \tag{2}$$

where $V[x(.)]$ is given in equation (1).

We say that $x(.)$ *is admissible* if if verifies the restrictions of the problem, i.e if $x(.) \in [x(t), t \in \mathbb{T} \; : \; x(0) = x_0, \text{ and } x(T) = x_T]$. We say that $x^*(.)$ *is optimal* if it is admissible and it maximizes the functional (1); therefore it is a solution of the problem (2).

Proposition 1. *Assume that $x^*(.)$ is optimal. Then it verifies the Euler equation*

$$F_x(t, x^*(t), \dot{x}^*(t)) - \frac{d}{dt}\left(F_{\dot{x}}(t, x(t), \dot{x}(t))\right) = 0. \tag{3}$$

and $x^*(0) = x_0$ and $x^*(T) = x_T$.

We denote $F_x(t, x^*(t), \dot{x}^*(t)) \equiv \left.\frac{\partial F(t,x(t),\dot{x}(t))}{\partial x}\right|_{x=x^*}$ and $F_{\dot{x}}(t, x(t), \dot{x}(t)) \equiv \frac{d}{dt}\left.\left(\frac{\partial F(t,x(t),\dot{x}(t))}{\partial \dot{x}}\right)\right|_{x=x^*}$.

Proof. Suppose that we know $x^*(.)$ and assume an admissible solution $x(.)$ such that $x(t) = x^*(t)+\epsilon h(t)$ for all $t \in \mathbb{T}$. If it is admissible then the increment should verify $h(0) = h(T) = 0$. The variation of the functional V is

$$\Delta V = V[x(.)] - V[x^*(.)] = \int_0^T \Big(F(t, x^*(t) + \epsilon h(t), \dot{x}^*(t) + \epsilon\dot{h}(t)) - F(t, x^*(t), \dot{x}^*(t))\Big)\, dt$$

Using the Taylor theorem, we get

$$\Delta V(\epsilon) = \int_0^T \Big(F_x(t, x^*(t), \dot{x}^*(t))\epsilon h(t) + F_{\dot{x}}(t, x^*(t), \dot{x}^*(t))\epsilon\dot{h}(t) + \ldots\Big)\, dt.$$

A functional derivative is defined as

$$\delta V = \lim_{\epsilon \to 0} \frac{\Delta V(\epsilon)}{\epsilon}$$

A necessary condition for an extremum of the functional is that $\delta V = 0$, for all admissible h. In our case, we have

$$\begin{aligned}
\delta V &= \int_0^T \Big(F_x(t, x^*(t), \dot{x}^*(t))h(t) + F_{\dot{x}}(t, x^*(t), \dot{x}^*(t))\dot{h}(t)\Big)\, dt = \\
&= \int_0^T F_x(t, x^*(t), \dot{x}^*(t))h(t)dt + F_{\dot{x}}(t, x^*(t), \dot{x}^*(t))h(t)\big|_{t=0}^{T} - \int_0^T \frac{d}{dt}F_{\dot{x}}(t, x^*(t), \dot{x}^*(t))h(t)dt = \\
&= \int_0^T \left(F_x(t, x^*(t), \dot{x}^*(t)) - \frac{d}{dt}F_{\dot{x}}(t, x^*(t), \dot{x}^*(t))\right) h(t)dt = 0
\end{aligned}$$

using integration by parts and the admissibility conditions $h(0) = h(T) = 0$. Using the fact that given two continuous functions $g(t)$ and $h(t)$ and if $\int_0^T g(t)h(t)dt = 0$ for every function $h(t)$ such that $h(0) = h(T) = 0$ then $g(t) = 0$ for every $t \in [0, T]$, then equation (3) results. □

Using the notation F_x^* and $F_{\dot{x}}^*$ for the first partial derivatives evaluated along the optimal path $x^*(.)$ and F_{xx}^*, $F_{\dot{x}\dot{x}}^*$ and $F_{x\dot{x}}^*$ for the second derivatives, observe that

$$\frac{d}{dt} F_{\dot{x}}^* = F_{\dot{x}t}^* + F_{\dot{x}x}^* \dot{x}^* + F_{\dot{x}\dot{x}}^* \ddot{x}^*$$

and the Euler equation can be written as a second order ordinary differential equation, if $F_{\dot{x}\dot{x}}^* \ddot{x}^* \neq 0$

$$F_x^* = F_{\dot{x}t}^* + F_{\dot{x}x}^* \dot{x}^* + F_{\dot{x}\dot{x}}^* \ddot{x}^*, \; 0 \leq t \leq T.$$

2.2 Canonical representation

Alternatively, the first order conditions can be presented under the so-called **canonical representation**. If $F_{\dot{x}\dot{x}} \ddot{x} \neq 0$ define the co-state function as

$$p(t) = F_{\dot{x}}(t, x(t), \dot{x}(t))$$

and the Hamiltonian function as

$$H(t, x(t), p(t)) = -F(t, x(t), \dot{x}(t)) + p(t)\dot{x}(t).$$

The Hamiltonian has the partial derivatives

$$H_x = \frac{\partial H}{\partial x} = -F_x, \; H_p = \frac{\partial H}{\partial p} = \dot{x}$$

because $H_{\dot{x}} = -F_{\dot{x}} + p = 0$. Then, the canonical representation of the Euler equation is

$$\dot{p} = -H_x(t, x, p), \; \dot{x} = H_p(t, x, p)$$

Therefore the solution for the simplest calculus of variation problem is (if $F(t,x,\dot{x})$ is concave in $(x,\dot{x})$) represented by the paths $(x^*(.), p(.))$ such that

$$\begin{cases} \dot{p} = -H_x(t, x^*(t), p(t)) & t \in [0,T] \\ \dot{x} = H_p(t, x^*(t), p(t)) & t \in [0,T] \\ x^*(0) = x_0 & t = 0 \\ x^*(T) = x_0 & t = T \end{cases} \tag{4}$$

Exercise: the cake eating problem The continous time version of the cake eating problem uses the discount factor $e^{-\rho t}$ where $\rho > 0$ is the rate of time preference and parameterizes impatience. The problem is to find the optimal flows of cake munching $C^*(.) = (C^*(t))_{t\in[0,T]}$ and of the size of the cake $W^*(.) = (W^*(t))_{t\in[0,T]}$ such that

$$\max_{C(.)} \int_0^T \ln(C(t)) e^{-\rho t} dt, \text{ subject to } \dot{W} = -C,\ t \in (0,T),\ W(0) = \phi,\ W(T) = 0 \tag{5}$$

where $\phi > 0$ is given. The problem can be equivalently written as a calculus of variations problem over the size of the cake $W(.)$:

$$\max_{W(.)} \int_0^T \ln(-\dot{W}(t)) e^{-\rho t} dt, \text{ subject to } W(0) = \phi,\ W(T) = 0$$

The Euler equation is

$$\ddot{W}^* + \rho \dot{W}^* = 0 \tag{6}$$

which is a second order linear ordinary differential equation.

We can use two alternative methods to find the optimal size of the cake $(W^*)_{t\in[0,T]}$

Method 1: two-step reduction to scalar differential equations Defining $z(t) = \dot{W}(t)$. Then we can transform equation (6) into a first order ODE

$$\dot{z}(t) = -\rho z(t)$$

which has the general solution

$$z(t) = ke^{-\rho t},\ t \in [0, T].$$

The inverse transformation $\dot{W}(t) = z(t)$ is a non-autonomous differential equation. We can solve it by the separation of variables method, by making $dW = z(t)dt$. Integrating and using the solution for $z(t)$ we get

$$\int_{W(0)}^{W(t)} dW = k \int_0^t e^{-\rho s} ds.$$

Performing the two integrations, we get

$$W(t) - W(0) = \frac{k}{\rho}\left(1 - e^{-\rho t}\right).$$

Using the initial and the terminal conditions $W(0) = \phi$ and $W(T) = 0$ we determine

$$k = -\frac{\rho}{1 - e^{-\rho T}}\phi.$$

Then the solution for the optimal size of the cake is

$$W^*(t) = \frac{e^{-\rho t} - e^{-\rho T}}{1 - e^{-\rho T}}\phi,\ t \in [0, T], \tag{7}$$

and for consumption, as $C^*(t) = -dW^*(t)/dt$,

$$C^*(t) = \rho\frac{e^{-\rho t}}{1 - e^{-\rho T}}\phi,\ t \in [0, T]. \tag{8}$$

Method 2: solving equation (6) **as a 1st order planar system** We can transform the second order ODE (6) into the first order planar linear ODE, $\dot{y} = Ay$, where $y = (y_1, y_2)^\top$ by making $y_1 = \dot{W}^*$ and $y_2 = \ddot{W}^* = \dot{y}_1$, and

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -\rho \end{pmatrix}$$

The solution is

$$\begin{aligned} y(t) &= Pe^{\mathbf{\Lambda} t}P^{-1}k = \\ &= \frac{1}{\rho}\begin{pmatrix} 1 & 1 \\ 0 & -\rho \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{-\rho t} \end{pmatrix}\begin{pmatrix} \rho & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}. \end{aligned}$$

As $W^*(t) = y_1(t)$ we get the general solution for W^*_t,

$$W^*(t) = k_1 + \frac{k_2}{\rho}\left(1 - e^{-\rho t}\right)$$

We use the initial and the terminal conditions $W(0) = \phi$ and $W(T) = 0$ to determine

$$k_1 = \phi,\ k_2 = -\frac{\rho}{1 - e^{-\rho T}}\phi.$$

And substituting back to $W^*(t)$, we get the optimal path for the cake size as in equation (7).

2.3 The simplest free endpoint problem

Now consider the problem: find a path $x^*(.) = (x^*(t))_{0 \le t \le T}$ such that

$$\begin{cases} \max_{x(.)} V[x(.)] \\ x(0) = x_0 \text{ given} \quad t = 0 \end{cases} \tag{9}$$

where $V[x(.)]$ is given in equation (1).

Proposition 2. *Assume that $x^*(.)$ is optimal. Then it verifies the Euler equation (3) the initial condition $x^*(0) = x_0$ and the transversality condition*

$$F_{\dot{x}}(t, x^*(T), \dot{x}(T)) = 0 \tag{10}$$

Proof. We use the same method of proof as in proposition 1, but we have now the variations $h(0) = 0$ and $h(T) \neq 0$, as $x^*(T)$ is free. Again $\delta V = 0$ is a necessary condition, but we get

$$\delta V = \int_0^T \left(F_x(t, x^*(t), \dot{x}^*(t)) - \frac{d}{dt}F_{\dot{x}}(t, x^*(t), \dot{x}^*(t))\right) h(t)dt + F_{\dot{x}}(t, x^*(T), \dot{x}^*(T))h(T) = 0$$

which implies that $F^*_x - \frac{d}{dt}F^*_{\dot{x}} = 0$ and $x^*(.)$ is extremal only if $F^*_{\dot{x}}(T)h(T) = 0$, but as in general $h(T) \neq 0$ then transversality condition (10) should hold. $\square$

2.4 Discounted infinite horizon problem

2.4.1 Free endpoint problem

The most common problem in macro-economics involve a infinite horizon discounted utility functional

$$V[x(.)] = \int_0^\infty f(x(t), \dot{x}(t))e^{-\rho t}dt,\ \rho > 0 \tag{11}$$

where $x(.) = (x(t))_{t\in[0,\infty)}$ and we assume $f(.)$ is continuous and $f(t,.)$ is smooth.

The calculus of variations problem is defined as: find $x^*(.) = (x(t))_{t\in[0,\infty)}$ such that

$$\begin{cases} \max_{x(.)} V[x(.)] \\ x(0) = x_0 \text{ given} \end{cases} \tag{12}$$

In this case, the Euler equation is a necessary condition and is, evaluated along the optimal trajectory $x^*(.)$

$$e^{-\rho t} f_x(x^*(t), \dot{x}^*(t)) - \frac{d}{dt}\left(e^{-\rho t} f_{\dot{x}}(x^*(t), \dot{x}^*(t))\right) = 0,\ \text{for } t \in [0,\infty)$$

or, if we evaluate the second derivative,

$$f_x(x^*(t), \dot{x}^*(t)) + \rho f_{\dot{x}}(x^*(t), \dot{x}^*(t)) - f_{\dot{x}x}(x, \dot{x})\dot{x}^* - f_{\dot{x}\dot{x}}(x^*(t), \dot{x}^*(t))\ddot{x}^* = 0,\ \text{for } t \in [0,\infty) \tag{13}$$

For the infinite horizon case, the transversality condition is not necessary in general. In order to solve for the two constants of integration, in addition to the initial condition $x(0) = x_0$, it is assumed that the solution tends to a steady state. In a steady state, $\overline{x}$, as $\dot{x} = \ddot{x} = 0$, we get in equation (13)

$$f_x(\overline{x}, 0) + \rho f_{\dot{x}}(\overline{x}, 0) = 0.$$

2.4.2 Calculus of variations: discounted infinite horizon constrained terminal value

The problem is defined as: find $x^*(.)$ verifying that

$$\begin{cases} \max_{x(.)} V[x(.)] \\ x(0) = x_0 \text{ given} \\ \lim_{t\to\infty} x(t) \geq 0 \end{cases} \tag{14}$$

The necessary conditions are:

$$\begin{cases} f_x(x^*, \dot{x}^*) + \rho f_{\dot{x}}(x^*, \dot{x}^*) - f_{\dot{x}x}(x^*, \dot{x}^*)\dot{x} - f_{\dot{x}\dot{x}}(x^*, \dot{x}^*)\ddot{x} = 0 \\ x^*(0) = \phi \\ \lim_{t\to\infty} e^{-\rho t} f_{\dot{x}}(x^*(t), \dot{x}^*(t))x^*(t) = 0 \end{cases}$$

2.4.3 Cake eating problem: infinite horizon

The problem is: find $W^*(.) = (W^*(t))_{t\in\mathbb{R}_+}$ that

$$\max_{W(.)} \int_0^\infty \ln(-\dot{W}(t))e^{-\rho t}dt$$

given $W(0) = \phi$ and $\lim_{t\to\infty} W(t) \geq 0$

The first order conditions are:

$$\begin{cases} \rho\dot{W}^*(t) + \ddot{W}^*(t) = 0 \\ W^*(0) = \phi \\ -\lim_{t\to\infty} e^{-\rho t}\frac{W^*(t)}{\dot{W}^*(t)} = 0 \end{cases}$$

We already found that the general solution from the first equation, after using the initial condition, is

$$W(t) = \phi - \frac{k}{\rho}\left(1 - e^{-\rho t}\right).$$

Then, because $\dot{W}(t) = -ke^{-\rho t}$

$$-\lim_{t\to\infty} e^{-\rho t}\frac{W^*(t)}{\dot{W}^*(t)} = \lim_{t\to\infty}\frac{\rho\phi - k\left(1 - e^{-\rho t}\right)}{ke^{-\rho t}}e^{-\rho t} = \frac{\rho\phi - k}{k} = 0$$

if and only if $k = \rho\phi$. Then the solution for the cake eating problem is

$$W^*(t) = \phi e^{-\rho t},\ t \in \mathbb{R}_+.$$

Bibliographic references Kamien and Schwartz (1991, part I)

3 Optimal control

The optimal control problem is a generalization of the calculus of variations problem. In addition to the state variable $x : \mathbb{T} \to \mathbb{R}$ we have a control variable $u : \mathbb{T} \to \mathbb{R}^m$ which controls the behavior of the state variable. Observe that we may have more than one control variable.

Now have the value functional is

$$V[u(.), x(.)] = \int_0^T f(u(t), x(t), t)dt$$

and the structure of the economy is represented by the ODE

$$\dot{x} = g(u(t), x(t), t), \ t \in [0, T]$$

given $x(0) = x_0$. The different versions of the problem vary according to differences in the horizon, T, in the terminal state of the economy $x(T)$ and in the existence of other constraints (that we will not study here).

3.1 The simplest problem

The simplest problem consists in finding the optimal paths $(x^*(.), u^*(.))$, where $x^*(.) \equiv (x^*(t))_{0 \le t \le T}$ and $u^*(.) \equiv (u^*(t))_{0 \le t \le T}$, which solve the problem:

$$\max_{u(.)} V[u(.), x(.)] = \max_{u(.)} \int_0^T f(u(t), x(t), t)dt \tag{15}$$

subject to

$$\begin{cases} \dot{x}(t) = & g(u(t), x(t), t) \\ x(0) = & x_0 \text{ given} \end{cases} \tag{16}$$

and $x(T)$ is free. The functions $(x(.), u(.))$ which verify conditions (16), for every $t \in [0, T]$ are called admissible solutions. Optimal solutions $(x^*(.), u^*(.))$ are admissible functions which allows us to get the optimal value function

$$V[x^*(.)] = \int_0^T f(u^*(t), x^*(t), t)dt = \max_{u(.)} V[u(.), x(.)].$$

Generally, while $x(t)$ continuous functions for every $t \in [0, T]$, $u(t)$ should be piecewise continuous in $t \in [0, T]$, but continuous it $t \in (0, T]$,

We call **Hamiltonian** to the function

$$H(t, x(t), u(t), \lambda(t)) \equiv f(t, x(t), u(t)) + \lambda(t)g(t, u(t), x(t)),\ t \in [0, T]$$

where function $\lambda : \mathbb{T} \to \mathbb{R}$ is called the co-state or adjoint variable.

The **maximized Hamiltonian** is

$$H^*(t, x(t), \lambda(t)) = f(t, x^*(t), u^*(t)) + \lambda(t)g(t, u^*(t), x^*(t)) = \max_{u(t)} H(t, x(t), u(t), \lambda(t)).$$

Theorem 1. The Pontriyagin's maximum principle
Let $(x^*(.), u^*(.))$ be a solution to the problem (15)-(16). Then there is a piecewise continuous function $\lambda(.)$ such that $(x^*(.), u^*(.), \lambda(.))$ simultaneously satisfy:

- *the admissibility condition:*

$$\begin{cases} \dot{x}^* = \frac{\partial H^*(t)}{\partial \lambda} = g(t, x^*(t), u^*(t)), & 0 < t \le T \\ x(0) = x_0, & t = 0 \end{cases} \tag{17}$$

- *the multiplier equation*

$$\begin{cases} \dot{\lambda} = -\frac{\partial H^*(t)}{\partial x} = -f_x(t, x^*(t), u^*(t)) - \lambda(t)g_x(t, x^*(t), u^*(t)), & 0 < t \le T \\ \lambda(T) = 0, & t = T \end{cases} \tag{18}$$

- *the optimality condition:*

$$\frac{\partial H^*(t)}{\partial u} = f_u(t, x^*(t), u^*(t)) + \lambda(t)g_u(t, x^*(t), u^*(t)) = 0,\ \ 0 \le t \le T. \tag{19}$$

Proof. This is just an heuristic proof using the variational principle. As $V[x^*(.)]$ is parameterized by x_0 then we can write $V^*(x_0) = V[x^*(.)]$. An optimal solution is an admissible solution which verifies

$$\begin{aligned} V^*(x_0) &= \int_0^T f(t, x^*(t), u^*(t))dt = \\ &= \int_0^T [f(t, x^*(t), u^*(t)) + \lambda g(t, x^*(t), u^*(t)) - \lambda(t)\dot{x}^*(t)]dt = \\ &= \int_0^T (H^*(t) - \lambda(t)\dot{x}^*(t))dt = \\ &= \int_0^T (H^*(t) + \dot{\lambda}(t)x^*(t))dt - \lambda(T)x^*(T) + \lambda(0)x^*(0) \end{aligned}$$

using integration by parts. Let us introduce a spike perturbation at time $t = 0$ from x_0 to $x_0 + a$. The solution of the perturbed problem has the value

$$V(x_0 + a) = \int_0^T (H(t) + \dot{\lambda}(t)x(t))dt - \lambda(T)x(t) + \lambda(0)(x(0) + a).$$

The variation of the value function is,

$$V(x_0+a)-V^*(x_0) = \int_0^T (H(t)+\dot{\lambda}(t)x(t)-H^*(t)-\dot{\lambda}(t)x^*(t))dt-\lambda(T)(x(T)-x^*(T))+\lambda(0)a.$$

If we perform a Taylor approximation and retain the first difference then

$$\begin{aligned} V(x_0 + a) - V^*(x_0) \approx & \int_0^T [(H^*_x(t) + \dot{\lambda}(t))(x(t) - x^*(t)) + H^*_u(t)(u(t) - u^*(t))]dt - \\ & -\lambda(T)(x(T) - x^*(T)) + \lambda(0)a. \end{aligned}$$

Then $\lim_{a\to 0}[V(x_0 + a) - V^*(x_0)] = 0$ if

$$H^*_x(t) + \dot{\lambda}(t) = H^*_u(t) = \lambda(T) = 0.$$

where

$$H^*_x(t) = \frac{\partial H(t, x^*(t), u^*(t), \lambda(t))}{\partial x}$$

and analogously for u. □

Observations:

1. The co-state variable can be interpreted as the marginal value of the state variable,

$$V_x(t, x(t)) = \lambda(t).$$

To prove this consider any initial time, t, and the associated optimal value for the state variable $x^*(t)$ as given and introduce a small perturbation to $x^*(t) + a$. Then from the previous proof we get

$$\frac{dV(t, x)}{dx} = \lim_{a \to 0} \frac{V(x^*(t) + a) - V(x^*(t))}{a} = \frac{\lambda(t)a}{a},$$

if along an optimal path. For $t = T$, we get $\lambda_T = V_x(x(T)) = 0$.

2. In the literature the terminal condition

$$V_x(x(T)) = \lambda(T) = 0$$

is called **transversality condition**. It has an obvious meaning: the terminal value of the state is chosen in a way that does not allow for marginal gains from changing it

3.2 Terminal state constraints

Now consider the problem

$$\max_{u(.)} V[u(.), x(.)] = \max_{u(.)} \int_0^T f(u(t), x(t), t)dt \tag{20}$$

subject to

$$\begin{cases} \dot{x} = g(u(t), x(t), t) & t \in (0, T) \\ x(0) = x_0 & t = 0 \\ h(T)x(T) \geq 0 \text{ given} & t = T \end{cases} \tag{21}$$

that is, we assume that the weighted terminal state is bounded below by zero.

The necessary first order conditions from the Pontryiagin's maximum principle are formally identical, with the exception of the constraint equation (18) which becomes

$$\lambda(T)x(T) = 0$$

In order to prove this, observe we introduce the Lagrangian, which becomes after performing the same method as in the proof of theorem (**??**)

$$L(x_0) = \int_0^T (H^*(t) + \dot{\lambda}(t)x^*(t))dt - \lambda(T)x^*(T) + \lambda(0)x^*(0) + \nu h(T)x^*(T)$$

where ν is a constant Lagrange multiplier associated to the terminal condition. Then

$$\begin{aligned} L(x_0 + a) \quad &- \quad L(x_0) = \\ &= \quad \int_0^T [(H_x^*(t) + \dot{\lambda}(t))(x(t) - x^*(t)) + \mathcal{H}_u^*(t)(u(t) - u^*(t))]dt \\ & \qquad -\lambda(T)(x(T) - x^*(T)) + \lambda(0)a + [\nu h(T) - \lambda(T)](x(T) - x^*(T)) \end{aligned}$$

Now, the optimality conditions are

$$H_x^*(t) + \dot{\lambda}(t) = H_u^*(t),\ t \in [0, T)$$

and

$$\nu h(T) - \lambda(T) = 0.$$

But as terminal state condition is an inequality, from the Kuhn-Tucker theorem the slackness conditions are

$$h(T)x(T) \geq 0, \ \ \nu \geq 0 \quad \text{and} \quad \nu h(T)x(T) = 0.$$

Therefore the transversality condition is $\lambda(T)x(T) = 0$ for any continuous function $h(T)$.

3.3 The discounted infinite horizon problem

Now, we will study the infinite horizon discounted problem, which has the utility functional

$$V[u(.), x(.)] = \int_0^\infty f(u(t), x(t))e^{-\rho t}dt, \ \rho > 0 \tag{22}$$

and the restriction (16).

The hamiltonian function is now called the **discounted** hamiltonian and is defined as

$$H(t, x(t), u(t), \lambda(t)) \equiv f(x(t), u(t))e^{-\rho t} + \lambda(t)g(u(t), x(t)), \ t \in \mathbb{R}_+$$

where $\lambda(.)$ is the discounted co-state variable.

It is convenient to use the **current value** hamiltonian h instead of the present value Hamiltonian H,

$$h(x(t), u(t), q(t)) = f(x(t), u(t)) + q(t)g(x(t), u(t)) = e^{\rho t}H(t, x(t), u(t), \lambda(t))$$

where the current value co-state variable is defined as

$$q(t) \equiv e^{\rho t}\lambda(t).$$

The current-value maximized hamiltonian is

$$h^*(x(t), q(t)) = \max_{u(t)} h(x(t), q(t), u(t)) \ t \in \mathbb{R}_+.$$

Passing the horizon to infinity comes with the cost of introducing some difficulties regarding the transversality condition, as compared with the finite horizon case. It ceases to be a necessary condition and becomes a sufficient condition. There are some ways to overcome this. One consists in imposing that the solution should converge to a steady state. Another, that we will consider next is to assume that functions $f(u, x)$ and $g(u, x)$ are concave (not necessarily strictly concave in the case of $g(.)$) in (u, v). In this case, The Pontriyagin's principle gives necessary and sufficient conditions for optimality.

Let $x(.) = (x(t))_{t\in\mathbb{R}_+}$ and $u(.) = (u(t))_{t\in\mathbb{R}_+}$ be admissible paths for the state and the control variables.

Theorem 2. The maximum principle of Pontryiagin for the infinite horizon discounted problem

Consider the optimal control problem (22)-(16) and assume that functions $f(u,x)$ and $g(u,x)$ are concave. Then $(x^*(.), u^*(.))$ is a solution of the optimal control problem if and only if there is piecewise continuous function $q(.)$ such that $(x^*(.), u^*(.), q(.))$ simultaneously satisfy:

- *the admissibility condition:*

$$\begin{cases} \dot{x}^* = \frac{\partial h^*}{\partial q} = g(x^*(t), u^*(t)), & t \in (0, +\infty) \\ x(0) = x_0, & t = 0; \end{cases} \tag{23}$$

- *the multiplier equation*

$$\begin{cases} \dot{q} = \rho q - \frac{\partial h^*}{\partial x} = \rho q(t) - f_x(x^*(t), u^*(t)) - q(t) g_x(x^*(t), u^*(t)), & t \in (0, +\infty); \\ \lim_{t\to\infty} e^{-\rho t} q(t) = 0 \end{cases} \tag{24}$$

- *and the optimality condition:*

$$\frac{\partial h^*}{\partial u} = f_u(x^*(t), u^*(t)) + q(t) g_u(x^*(t), u^*(t)) = 0, \quad t \in [0, +\infty]. \tag{25}$$

If we assume that there is a terminal condition as

$$\lim_{t\to\infty} h(t)x(t) \geq 0$$

the terminal condition analogous to equation (24) becomes

$$\lim_{t\to\infty} e^{-\rho t} q(t) x(t) = 0.$$

3.4 The hamiltonian dynamic system

We have been assuming that functions $f(.)$ and $g(.)$ are continuous and differentiable as regards (u, x). If in addition assume that $\partial^2 h(u, x)/\partial u^2 \neq 0$ (or $\det(h_{uu}) \neq 0$ if there is more than one control variable). Then, from the implicit function theorem, we can determine locally, from the optimality condition (25)

$$u^*(t) = u^*(x(t), q(t)),\ t \in \mathbb{R}_+$$

where

$$\frac{\partial u^*}{\partial q} = -h_{uu}^{-1} g_u, \quad \frac{\partial u^*}{\partial x} = -h_{uu}^{-1} h_{ux},$$

where

$$h_{uu} = f_{uu} + q g_{uu},\ h_{uq} = g_u,\ h_{ux} = f_{ux} + q g_{ux}.$$

Then

$$h_x^*(x, q) = f_x(x^*, u^*(x^*, q)) - q g_x(x^*, u^*(x^*, q))$$

and

$$h_q^*(x, q) = f(x^*, u^*(x^*, q)).$$

If we substitute the control variables into the differential equations in (23) -(24) we get a mixed initial-terminal value problem for the planar ordinary differential equation which is called the **modified hamiltonian dynamic system**

$$\dot{q} = \rho q(t) - h_x(x^*(t), q(t)) \quad (26)$$

$$\dot{x}^* = h_q(x^*(t), q(t)) \quad (27)$$

together with the initial condition $x^*(0) = x_0$ and the transversality condition $\lim_{t\to+\infty} e^{-\rho t} q(t) = 0$ or $\lim_{t\to+\infty} e^{-\rho t} q(t) x^*(t) = 0$.

Proposition 3. Local dynamics for the MhDS

Let functions $f(.)$ and $g(.)$ be continuous and differentiable. Then the local stable manifold *of the MHDS has dimension 1 if and only if the determinant of its Jacobian evaluated in the neighborhood of its fixed point is negative.*

Proof. Let $y(t) = (p(t), x(t))$ and assume that the MHDS has a fixed point $\overline{y}$. Then the MHDS may be approximated, under certain conditions, in the neighborhood of $\overline{y}$ by the linear system $\dot{y}(t) = \mathbb{J}(y(t) - \overline{y})$, where $\mathbb{J} = D_y(G(\overline{y}))$. If $\det(h_{uu}) \neq 0$ we see that the jacobian is of type

$$\mathbb{J} = \begin{pmatrix} \dot{q} \\ \dot{x}^* \end{pmatrix} = \begin{pmatrix} \rho - a & b \\ c & a \end{pmatrix}$$

where

$$\begin{aligned} a &:= -\frac{g_u h^*_{ux}}{h^*_{uu}} \\ b &:= -\left(\frac{h^*_{xx}h^*_{uu} - (h^*_{ux})^2}{h^*_{uu}}\right) \\ c &:= -\frac{(g_u)^2}{h^*_{uu}} \end{aligned}$$

if $m = 1$ or

$$\begin{aligned} a &:= -h^*_{xu}u^*_q = h^*_{xu}h^{-1}_{uu}h_{uq} = h^*_{qu}(h^{-1}_{uu})^T h_{ux} \\ b &:= -\left(h^*_{xx} - h^*_{xu}(h^*_{uu})^{-1}h^*_{ux}\right) \\ c &:= -h^*_{qu}(h^*_{uu})^{-1}h^*_{uq} \end{aligned}$$

if $m > 1$, as $h_{xu} = h^T_{ux}$ and $h_{qu} = h^T_{uq}$ (from the continuity of functions $f(.)$ and $g(.)$)are $(1 \times m)$-vectors and h_{xx} is non-singular.

Then we will have

$$\begin{aligned} \mathrm{tr}(\mathbf{J}) &= \rho > 0 \\ \det(\mathbf{J}) &= a(\rho - a) - bc \end{aligned}$$

□

It is natural that the optimal trajectory should be a saddle path: given the initial value of the state variable, $x(0)$, the optimal path that verifies the first order conditions is unique for u and x.

Bibliographic references Kamien and Schwartz (1991, part II), Chiang (1992). The original presentation is in Pontryagin et al. (1962)

3.4.1 Application: the cake eating problem

Consider, again, problem (5). The current value hamiltonian is

$$h(t) = \ln\left(C(t)\right) - q(t)C(t)$$

where q is the co-state variable. The first order conditions are

$$\begin{aligned}
\frac{\partial h^*(t)}{\partial C(t)} &= C^*(t)^{-1} - q(t) = 0, \\
\dot{q} &= \rho q(t) - \frac{\partial h^*}{\partial W} = \rho q(t) \\
\dot{W}^* &= \frac{\partial h^*}{\partial q} = -C^*(t) \\
W^*(0) &= \phi \\
W^*(T) &= 0.
\end{aligned}$$

As $\dot{C}^*/C^* = -\dot{q}/q$ from the first equation, we get an equivalent problem

$$\begin{aligned}
\dot{C}^* &= -\rho C^*(t) \\
\dot{W}^* &= -C^*(t) \\
W^*(0) &= \phi \\
W^*(T) &= 0.
\end{aligned}$$

The two first equations form a linear planar ODE of type $\dot{y} = Ay$, where

$$A = \begin{pmatrix} -\rho & 0 \\ -1 & 0 \end{pmatrix}.$$

The solution is

$$\begin{pmatrix} C^*(t) \\ W^*(t) \end{pmatrix} = \frac{1}{\rho}\begin{pmatrix} 0 & \rho \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{-\rho t} \end{pmatrix}\begin{pmatrix} -1 & \rho \\ 1 & 0 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = $$
$$= \begin{pmatrix} -k_1 e^{-\rho t} \\ k_2 - \frac{1}{\rho}\left(1 - e^{-\rho t}\right) \end{pmatrix}.$$

Using the initial and the terminal conditions, $W(0) = \phi$ and $W(T) = 0$, we get the same solutions as for the calculus of variation version, equations (8)-(7).

3.4.2 Application: Resource depletion problem

Assume there is a non-renewable resource where $N(t)$ denotes its endowment at time t and that the resource is depleted by consumption. What would be the optimal rate of depletion if in the economy there is a representative agent who maximizes the discounted intertemporal path of consumption and we assume that the asymptotic value of the resource cannot be negative.

The problem is

$$\max_{C(.)} \int^{\infty} e^{-\rho t} \ln\left(C(t)\right) dt, \; \rho > 0$$

subject to

$$\begin{cases} \dot{N}(t) = -C(t), \; t \in [0, \infty) \\ N(0) = N_0, \text{ given} \\ \lim_{t\to\infty} N(t) \geq 0 \end{cases}$$

The current-vale hamiltonian is

$$h(t) = ln(C(t)) - q(t)C(t)$$

and the first order conditions are:

$$\begin{aligned} C(t) &= 1/q(t) \\ \dot{q} &= \rho q(t) \\ \lim_{t\to\infty} e^{-\rho t} q(t) N(t) &= 0 \\ \dot{N} &= -C(t) \\ N(0) &= N_0 \end{aligned}$$

If we differentiate the optimality condition and substitute the Euler equation we get the MHDS

$$\begin{aligned} \dot{C} &= -\rho C(t) \\ \dot{N} &= -C(t) \\ N(0) &= N_0 \\ \lim_{t\to\infty} e^{-\rho t} \frac{N(t)}{C(t)} &= 0 \end{aligned}$$

We can solve the MHDS in two steps:

- 1st step: we define $z(t) \equiv N(t)/C(t)$ and consider the transversality condition

$$\begin{cases} \dot{z} = -1 + \rho z \\ \lim_{t\to\infty} e^{-\rho t} z(t) = 0 \end{cases}$$

  the solution is constant

$$z(t) = \frac{1}{\rho},\ t \in [0, \infty)$$

- 2nd step: we substitute the solution for z in

$$\begin{cases} \dot{N} = -C(t) = -N(t)/z(t) \\ N(0) = N_0 \end{cases}$$

and solve for $N(.)$ to get the solution for the resource endowment

$$N^*(t) = N_0 e^{-\rho t},\ t \in [0, \infty).$$

Characterization of the solution:

- there is asymptotic extinction
$$\lim_{t\to\infty} N^*(t) = 0$$
- the speed of adjustment can be assessed by computing the half-life of the process
$$\tau \equiv \left\{ t:\ N^*(t) = \frac{N(0) - N^*(\infty)}{2} \right\} = -\frac{\ln{(1/2)}}{\rho}$$
if $\rho = 0.02$ then $\tau \approx 34.6574$ years

3.4.3 Application: The Ramsey problem

The Ramsey problem is:

$$\max_C \int^{\infty} e^{-\rho t} U(C(t))dt,\ \rho > 0,$$

subject to

$$\dot{K}(t) = F(K(t)) - C(t),\ t \in [0, \infty)$$

and $K(0) = K_0$ given and $\lim_{t\to\infty} e^{-\rho t} K(t) \geq 0$.

We assume that $u(C)$ and $F(K)$ are Increasing, concave and Inada:

$$U^{'}(.) > 0,\ U^{''}(.) < 0, F^{'}(.) > 0,\ F^{''}(.) < 0$$

$$U^{'}(0) = \infty,\ U^{'}(\infty) = 0,\ F^{'}(0) = \infty,\ F^{'}(\infty) = 0$$

The current-value Hamiltonian is

$$h(C(t), K(t), Q(t)) = U(C(t)) + Q(t)(F(K(t)) - C(t))$$

The first order conditions according to the Pontriyagin's principle are:

$$
\begin{aligned}
U^{'}(C(t)) &= Q(t),\ t \in \mathbb{R}_+ \\
\dot{Q} &= Q(t)\left(\rho - F^{'}(K(t))\right),\ t \in \mathbb{R}_+ \\
\lim_{t\to\infty} e^{-\rho t}Q(t)K(t) &= 0 \\
\dot{K} &= F(K(t)) - C(t) \\
K(0) &= K_0
\end{aligned}
$$

The MHDS is

$$
\begin{aligned}
\dot{C} &= \frac{C(t)}{\sigma(C(t))}\left(F^{'}(K(t)) - \rho\right) \\
\dot{K} &= F(K(t)) - C(t) \\
K(0) &= K_0 > 0 \\
0 &= \lim_{t\to\infty} e^{-\rho t}U^{'}(C(t))K(t)
\end{aligned}
$$

where

$$
\sigma(C) \equiv -\frac{U''(C)C}{U'(C)}
$$

is the elasticity of intertemporal substitution.

As we do not specified functions $U(.)$ and $F(.)$, the MHDS has no explicit solution. But we can use a qualitative approach in order to characterise the solution.

In order to do it, we first determine the steady state(s) $(\bar{C}, \bar{K})$, linearize the MDHS in the neighbourhood of the steady states, check if the transversality condition holds, and then characterise the linearised dynamics in the neighbourhood of an admissible steady state.

The steady state (if $K > 0$)

$$\begin{aligned} F^{'}(\bar{K}) &= \rho \Rightarrow \bar{K} = (F^{'})^{-1}(\rho) \\ \bar{C} &= F(\bar{K}) \end{aligned}$$

Linearized system

$$\begin{pmatrix} \dot{C} \\ \dot{K} \end{pmatrix} = \begin{pmatrix} 0 & \frac{\bar{C}}{\sigma(\bar{C})} F^{''}(\bar{K}) \\ -1 & \rho \end{pmatrix} \begin{pmatrix} C(t) - \bar{C} \\ K(t) - \bar{K} \end{pmatrix}$$

The jacobian $\mathbf{J}$ has trace and determinant:

$$\mathrm{tr}(\mathbf{J}) = \rho,\ \det(\mathbf{J}) = \frac{\bar{C}}{\sigma(\bar{C})} F^{''}(\bar{K}) < 0$$

then the discriminant is

$$\Delta = \left(\frac{\rho}{2}\right)^2 - \frac{\bar{C}}{\sigma(\bar{C})} F^{''}(\bar{K}) > \left(\frac{\rho}{2}\right)^2 > 0$$

The eigenvalues of $\mathbf{J}$ are

$$\lambda^s = \frac{\rho}{2} - \sqrt{\Delta} < 0,\ \lambda^u = \frac{\rho}{2} + \sqrt{\Delta} > \rho$$

then $(\bar{C}, \bar{K})$ is a saddle point.

The eigenvector matrix is

$$\mathbf{V} = (V^s V^u) = \begin{pmatrix} \lambda^u & \lambda^s \\ 1 & 1 \end{pmatrix}.$$

The optimal solution $(C^*(.), K^*(.))$ is tangent, asymptotically, to the local stable manifold:

$$\begin{pmatrix} C^*(t) \\ K^*(t) \end{pmatrix} = \begin{pmatrix} \bar{C} \\ \bar{K} \end{pmatrix} + K_0 \begin{pmatrix} \lambda^u \\ 1 \end{pmatrix} e^{\lambda^s t},\ t \in [0, \infty).$$

This means that the local stable manifold has slope higher than the isocline $\dot{K}(C, K) = 0$

$$\left.\frac{dC}{dK}\right|_{W^s} (\overline{C}, \overline{K}) = \lambda^u > \left.\frac{dC}{dK}\right|_{\dot{K}} (\overline{C}, \overline{K}) = F^{'}(\bar{K}) = \rho$$

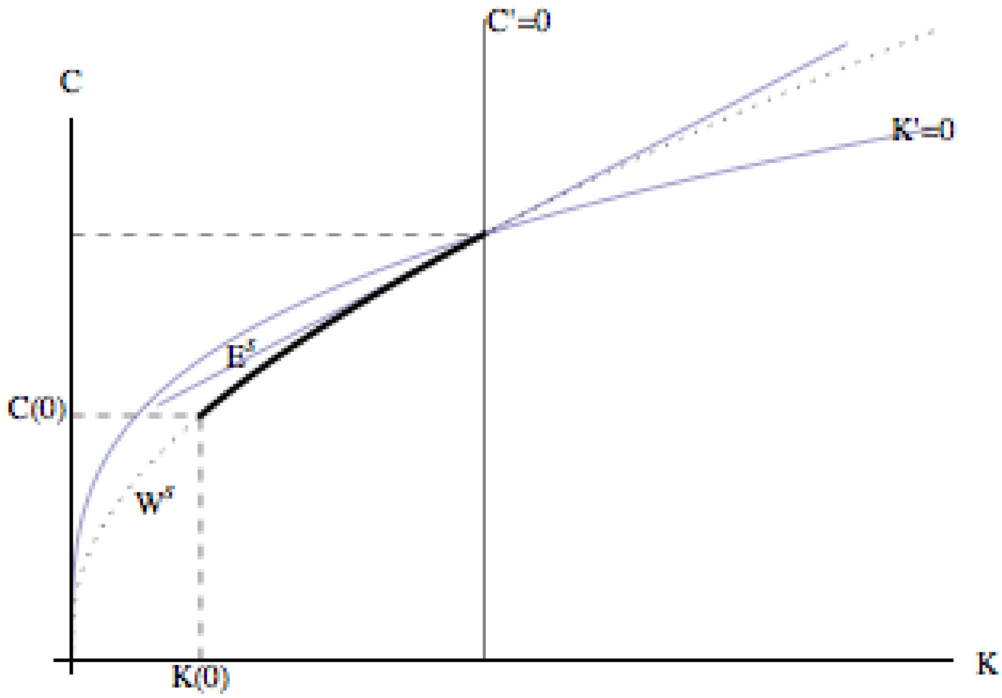

Figure 1: Ramsey model: phase diagram

4 Dynamic programming

4.1 Simplest problem

In the space of the functions $(u(.), x(.)) = (u(t), x(t))_{t\in[t_0,t_1]}$ for $t_0 \leq t \leq t_1$ find functions $(u^*(.), x^*(.))$ which solve the problem:

$$\max_{u(.)} \int_{t_0}^{t_1} f(t, x(t), u(t))dt$$

subject to

$$\dot{x} \equiv \frac{dx(t)}{dt} = g(t, x(t), u(t)),\ t_0 \leq t \leq t_1$$

given $x(t_0) = x_0$. We assume that t_1 is know and that $x(t_1)$ is free.

The value function is, for the initial instant

$$\mathcal{V}(t_0, x_0) = \int_{t_0}^{t_1} f(t, x^*(t), u^*(t))dt$$

and for the terminal time $\mathcal{V}(t_1, x(t_1)) = 0$.

Lemma 1. First order necessary conditions for optimality from the Dynamic Programming principle
Let $\mathcal{V} \in C^2(\mathbb{T}, \mathbb{R})$. Then the value function which is associated to the optimal path $((x^*(t), u^*(t))_{t_0\leq t\leq t_1}$ verifies the fundamental partial differential equation or the **Hamilton-Jacobi-Bellman equation**

$$-\mathcal{V}_t(t, x) = \max_u [f(t, x, u) + \mathcal{V}_x(t, x)g(t, x, u)].$$

Proof. Consider the value function

$$
\begin{aligned}
\mathcal{V}(t_0,x_0) &= \max_{(u(t))_{t_0\le t\le t_1}} \left(\int_{t_0}^{t_1} f(t,x(t),u(t))dt \right) \\
&= \max_{(u(t))_{t_0\le t\le t_1}} \left(\int_{t_0}^{t_0+\Delta t} f(.)dt + \int_{t_0+\Delta t}^{t_1} f(.)dt \right) = \quad (\text{for } \Delta t>0, \text{ small}) \\
&= \max_{(u(t))_{t_0\le t\le t_0+\Delta t}} \left[\int_{t_0}^{t_0+\Delta t} f(.)dt + \max_{(u(t))_{t_0+\Delta t\le t\le t_1}} \left(\int_{t_0+\Delta t}^{t_1} f(.)dt \right) \right] = \\
&\quad (\text{from dynamic prog principle}) \\
&= \max_{(u(t))_{t_0\le t\le t_0+\Delta t}} \left[\int_{t_0}^{t_0+\Delta t} f(.)dt + \mathcal{V}(t_0+\Delta t, x_0+\Delta x) \right] = \\
&\quad (\text{approximating } x(t_0+\Delta t)\approx x_0+\Delta x) \\
&= \max_{(u(t))_{t_0\le t\le t_0+\Delta t}} [f(t_0,x_0,u)\Delta t + \mathcal{V}(t_0,x_0) + \mathcal{V}_t(t_0,x_0)\Delta t + \mathcal{V}_x(t_0,x_0)\Delta x + \text{h.o.t}]
\end{aligned}
$$

if $u \approx$ constant and $\mathcal{V} \in C^2(\mathbb{T},\mathbb{R})$). Passing $\mathcal{V}(t_0,x_0)$ to the second member, dividing by Δt and taking the limit $\lim_{\Delta t\to 0}$ we get, for every $t \in [t_0,t_1]$,

$$0 = \max_u [f(t,x,u) + \mathcal{V}_t(t,x) + \mathcal{V}_x(t,x)\dot{x}].$$

□

Observations:

- In the DP theory the function $u^* = h(t,x)$ is called the **policy function** . Then the HJB equation may be written as

$$-\mathcal{V}_t(t,x) = f(t,x,h(t,x)) + \mathcal{V}_x(t,x)g(t,x,h(t,x))].$$

- Though the differentiability of $\mathcal{V}$ is assured for the functions f and g which are common in the economics literature, we can get explicit solutions, for $V(.)$ and for $h(.)$, only in very rare cases. Proving that $\mathcal{V}$ is differentiable, even in the case in which we cannot determine it explicitly is hard and requires proficiency in Functional Analysis.

Relationship with the Pontriyagin's principle:

(1) If we apply the transformation $\lambda(t) = \mathcal{V}_x(t, x(t))$ we get the following relationship with the Hamiltonian function which is used by the Pontriyagin's principle: $-\mathcal{V}_t(t, x) = H^*(t, x, \lambda)$;

(2) If $\mathcal{V}$ is sufficienty differentiable, we can use the principle of DP to get necessary conditions for optimality similar to the Pontriyagin principle.
The maximum condition is

$$f_u + \mathcal{V}_x g_u = f_u + \lambda g_u = 0$$

and the canonical equations are: as $\dot{\lambda} = \frac{\partial \mathcal{V}_x}{\partial t} = \mathcal{V}_{xt} + \mathcal{V}_{xx} g$ and differenting the HJB as regards x, implies $-\mathcal{V}_{tx} = f_x + \mathcal{V}_{xx} g + \mathcal{V}_x g_x$, therefore the canonical equation results

$$-\dot{\lambda} = f_x + \lambda g_x.$$

(3) Differently from the Pontryiagin's principle which defines a dynamic system of the form $(\mathbb{T}, \mathbb{R}^2, \varphi_t = (q(t), x(t)))$, the principle of dynamic programming defines a dynamic system as $((\mathbb{T}, \mathbb{R}), \mathbb{R}, v_{t,x} = \mathcal{V}(t, x)))$. That is, if defines a recursive mechanism in all or in a subset of the state space.

4.2 Infinite horizon discounted problem

Lemma 2. First order necessary conditions for optimality from the Dynamic Programming principle
Let $\mathcal{V} \in C^2(\mathbb{T}, \mathbb{R})$*. Then the value function associated to the optimal path* $((x^*(t), u^*(t))_{t_0 \le t < +\infty}$

verifies the fundamental non-linear ODE called the **Hamilton-Jacobi-Bellman equation**

$$\rho V(x) = \max_{u}[f(x,u) + V'(x)g(x,u)].$$

Proof. Now, we have

$$\begin{aligned} \mathcal{V}(t_0, x_0) &= \max_{u(.)} \left(\int_{t_0}^{+\infty} f(x,u)e^{-\rho t} dt \right) = \\ &= e^{-\rho t_0} \max_{u(.)} \left(\int_{t_0}^{+\infty} f(x,u)e^{-\rho(t-t_0)} dt \right) = \\ &= e^{-\rho t_0} V(x_0) \end{aligned} \tag{28}$$

where $V(.)$ is independent from t_0 and only depends on x_0. We can do

$$V(x_0) = \max_{u(.)} \left(\int_{0}^{+\infty} f(x,u)e^{-\rho t} dt \right).$$

If we let, for every (t,x), $\mathcal{V}(t,x) = e^{-\rho t}V(x)$ and if we substitute the derivatives in the HJB equation for the simplest problem, we get the new HJB. □

Observations:

- if we determine the policy function $u^* = h(x)$ and substitute in the HJB equation, we see that the new HJB equation is a ODE of the type $\dot{x}(t) = a(t) + b(t)x(t)$. This new HJB defines a recursion over x. Intuitively it generates a rule which says : if we observe the state x the optimal policy is $h(x)$ in such a way that the initial value problem should be equal to the present value of the variation of the state.

- It is still very rare to find explicit solutions for $V(x)$. There is a literature on how to compute it numerically, which is related to the numerical solution of ODE's and not with approximating value functions as in the discrete time case.

Bibliographic references Kamien and Schwartz (1991, part II, section 21) Grass et al. (2008)

Application: the cake eating problem Consider again problem (5). Now, we want to solve it by using the principle of the dynamic programming. In order to do it, we have to determine the value function $V = V(t, W)$ which solves the HJB equation

$$-\frac{\partial V}{\partial t} = \max_{C(.)} \left\{ e^{-\rho t} \ln(C) - C \frac{\partial V}{\partial W} \right\}$$

The optimal policy for consumption is determined from

$$C^*(t) = e^{-\rho t} \left(\frac{\partial V}{\partial W} \right)^{-1}$$

If we substitute back into the HJB equation we get the partial differential equation

$$-e^{\rho t} \frac{\partial V}{\partial t} = \ln \left[e^{-\rho t} \left(\frac{\partial V}{\partial W} \right)^{-1} \right] - 1$$

To solve it, let us use the method of determined coefficients by conjecturing that the solution is of the type

$$V(t, W) = e^{-\rho t}(a + b \ln W)$$

where a and b are constants to be determined, if our conjecture is right. With this function, the HJB equation comes

$$\rho(a + b \ln W) = \ln(W) - \ln b - 1$$

if we set $b = 1/\rho$ we eliminate the term in $\ln W$ and get

$$a = -(1 - \ln(\rho))/\rho.$$

Therefore, solution for the HJB equation is

$$V(t, W) = \frac{-1 + \ln(\rho) + \ln W}{\rho} e^{-\rho t}$$

and the optimal policy for consumption is

$$C^*(t) = \rho W(t).$$

References

Alpha Chiang. *Elements of Dynamic Optimization*. McGraw-Hill, 1992.

Dieter Grass, Jonathan P. Caulkins, Gustav Feichtinger, Gernot Tragler, and Doris A. Behrens. *Optimal Control of Nonlinear Processes. With Applications in Drugs, Corruption, and Terror*. Springer, 2008.

Morton I. Kamien and Nancy L. Schwartz. *Dynamic optimization, 2nd ed.* North-Holland, 1991.

L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko. *The Mathematical Theory of Optimal Processes*. Interscience Publishers, 1962.