Probability and Stochastic Processes

Master in Actuarial Sciences

Alexandra Bugalho de Moura

2018/2019

Introduction to Copulas

Introduction to Copulas

Multivariate Models

- We will now adress the issue of possible dependencies between risks
- All information about the relationship between random variables is captured by the multivariate distribution
- We are interested in building bivariate or multivariate models from (possibly different) known marginal distributions and a dependence between risks

Tail dependence

We are particularly interested in understanding dependencies between r.v. in the tail, *i.e.* when very large losses occur

- " If one risk has a large loss, is it more likely that another risk will also have a large loss?"
- "What are the odds of having several large losses from different risk types?"

Introduction to Copulas

Multivariate Models

- Historically, many multivariate distributions have been developed as immediate extensions of univariate distributions (e.g. multivariate normal distribution; bivariate Pareto).
- The drawbacks of these types of distributions are that
    1. one needs a different family for each marginal distribution
    2. extensions to more than just the bivariate case are not clear
    3. measures of association often appear in the marginal distributions
- A construction of multivariate distributions not suffering from these drawbacks is based on the *Copula* function

Introduction to Copulas

Example: bivariate Pareto distribution

- Consider two r.v. such that $X_1|\Lambda = \lambda \sim Exp(1/\lambda)$ and $X_2|\Lambda = \lambda \sim Exp(1/\lambda)$, with $\Lambda \sim Gamma(\alpha, 1/\theta)$.
- We known that $X_1 \sim Pareto(\alpha, \theta)$ and $X_2 \sim Pareto(\alpha, \theta)$
- Since X_1 and X_2 share the same random effect, they are not independent
- What is the joint cdf of (X_1, X_2)?

Remembering that

$$F_{X_1,X_2}(x_1, x_2) = P(X_1 \leqslant x_1, X_2 \leqslant x_2) = F_1(x_1) + F_2(x_2) - 1 + P(X_1 > x_1, X_2 > x_2)$$

and

$$P(X_1 > x_1, X_2 > x_2) = \int_0^{+\infty} P(X_1 > x_1, X_2 > x_2|\Lambda = \lambda)\, f_\Lambda(\lambda)\, d\lambda = \int_0^{+\infty} e^{-\lambda(x_1+x_2)}\, f_\Lambda(\lambda)\, d\lambda$$

(given $\Lambda = \lambda$, X_1 and X_2 are independent, as the random effect of Λ does not exist anymore)

Then

$$F_{X_1,X_2}(x_1, x_2) = F_1(x_1) + F_2(x_2) - 1 + \left[(1 - F_1(x_1))^{-1/\alpha} + (1 - F_2(x_2))^{-1/\alpha} - 1\right]^{-\alpha}$$

Introduction to Copulas

History

- **1951** Fréchet problem: given the marginal distribution functions, what can be said about the multivariate distribution?

Maurice Fréchet (1878 – 1973)

Introduction to Copulas

History

- **1959** Sklar introduces the notion and name *Copula*
- In his paper he introduces the most important result in this respect: multivariate cumulative distribution functions can be expressed in terms of copulas

Abe Sklar

Introduction to Copulas

History

- **1981** Schweizer & Wolff paper relating copulas to the study of dependence amongst r.v.

"Quite by accident, reread a paper by A. Rényi, entitled *On measures of dependence*, and realized that one could easily construct such measures by using copulas" (Berthold Schweizer)

"After the publication of these articles and of the book (*Probabilistic Metric Spaces*, Schweizer & Sklar, 1974) the pace quickened, as more students and colleagues became involved"

History

- **1990's** Books from Joe (1997), *Multivariate Models* and Nelsen (1999), *Introduction to Copulas*
- The notion of copulas is discovered by researchers in several applied fields

Introduction to Copulas

History

- **Nowadays** Widely used in many fields of applications, namely (but not only) finance, risk management and actuarial sciences

"The notion of copula is both natural as well as easy for looking at multivariate df's. But why do we witness such an incredible growth in papers published starting the end of the 1990's (recall, the concept goes back to the 1950's and even earlier, but not under that name)? Here I can give three reasons: finance, finance, finance. In the 1980's and 1990's we experienced an explosive development of quantitave risk management methodology within finance and insurance, a lot of which was driven by either new regulatory guidelines or the development of new products." (Paul Embrechts)

"The era of i.i.d. is over: and when dependence is taken seriously, copulas naturally come into play." (Carlo Sempi)

Introduction to Copulas

Copulas

- When two risks are assumed not to be independent, an infinite range of possible dependencies between them can be at stake.
- The first question is, if they are dependent, what is the best model to explain the existing dependencies
- Copulas constitute a convenient and elegant way of describing dependencies between two or more random variables

Copulas

- The joint distribution function is expressed as a parametric function of the marginal distribution functions.
- The joint probability function is decomposed into the marginal probability functions and a dependence structure component.
- Not only the joint distribution is known through the margins, as the dependence structure is decoupled from them.
- Being parametric functions of the margins, the copula offers a natural procedure for the estimation of the multivariate distribution simply by plugging in the evaluation of each marginal.

Sklar's theorem and copulas

The bivariate case

- Let X_1 and X_2 be r.v. with distribution functions $F_1(x_1)$ and $F_1(x_2)$
- A copula, $C(u_1, u_2)$, is a function that maps $(u_1, u_2) = (F_1(x_1), F_2(x_2))$, i.e. the unit square $[0,1] \times [0,1]$, into $[0,1]$ which will be the value of the joint distribution function:

$$\begin{aligned} C: \quad [0,1] \times [0,1] &\rightarrow [0,1] \\ (F_1(x_1), F_2(x_2)) &\mapsto F_{\mathbf{X}}(x_1, x_2) = P(X_1 \leqslant x_1, X_2 \leqslant x_2) = C(F_1(x_2), F_2(x_2)) \end{aligned}$$

- Equivalently, a copula is a restriction to the unit square $[0,1]^2$ of a bivariate distribution function whose margins are uniform in $[0,1]$:

$$U_1 = F_1(X_1) \sim U(0,1) \quad \text{and} \quad U_2 = F_2(X_2) \sim U(0,1)$$

and the distribution function of (U_1, U_2) is a copula:

$$\begin{aligned} C(u_1, u_2) &= P(U_1 \leqslant u_1, U_2 \leqslant u_2) = P(X_1 \leqslant F_1^{-1}(u_1), X_2 \leqslant F_2^{-1}(u_2)) \\ &= F_{\mathbf{X}}(F_1^{-1}(u_1), F_2^{-1}(u_2)) = F_{\mathbf{X}}(x_1, x_2) \end{aligned}$$

Sklar's theorem and copulas

Definition: bivariate copula

A bivariate copula, C, is a non-decreasing and right-continuous function, mapping $[0,1] \times [0,1]$ into $[0,1]$ such that, for all (u_1, u_2)

i) $\lim_{u_1 \to 0} C(u_1, u_2) = 0$ and $\lim_{u_2 \to 0} C(u_1, u_2) = 0$ (C is grounded)

ii) $\lim_{u_1 \to 1} C(u_1, u_2) = u_2$ and $\lim_{u_2 \to 1} C(u_1, u_2) = u_1$ (C has margins)

ii) C is supermodular or 2-increasing:

$$C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) + C(u_1, v_1) \geqslant 0, \quad \text{for} \quad u_1 \leqslant u_2 \quad \text{and} \quad v_1 \leqslant v_2$$

Remarks

- $u_i = F_i(x_i)$
- $C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) + C(u_1, v_1)$ is the so-called C-volume of $[u_1, v_1] \times [u_2, v_2]$
- These 3 properties ensure that the copula correctly capture the properties one would expect of a joint distribution of X_1 and X_2 in all circumstances:

1. $C(F_1(x_1), F_2(x_2))$ is a legitimate multivariate distribution function
2. The marginal distributions match those of X_1 and X_2

Sklar's theorem and copulas

The multivariate case

The definition of copula can be extended to the multivariate case.

- Let $X_1, X_2, \ldots, X_d$ be r.v. with distribution functions $F_1(x_1), F_2(x_2), \ldots, F_d(x_d)$

A multivariate copula, C, is a non-decreasing and right-continuous function, mapping $[0,1]^d$ into $[0,1]$ such that, for all $(u_1, u_2, \ldots, u_d)$

i) $C(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_d) = 0, \quad i = 1, 2, \ldots, d$ (C is grounded)

ii) $C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i, \quad i = 1, 2, \ldots, d$ (C has margins)

ii) C-volume$([\boldsymbol{u}, \boldsymbol{v}]) \geqslant 0$, for $[\boldsymbol{u}, \boldsymbol{v}] = [u_1, v_1] \times \cdots \times [u_d, v_d]$ (C is d-increasing)

Here we will focus on bivariate copulas, or, equivalently, on dependency structures between pairs of random variables.

Sklar's theorem and copulas

The Sklar's theorem clarifies the role of copulas in associating multivariate and marginal distribution functions

Sklar's theorem

Let X_1 and X_2 be random variables with distribution functions $F_1(x_1)$ and $F_2(x_2)$, respectively. Then, there exists a copula C such that, for all $(x_1, x_2) \in \mathbb{R}^2$

$$F_{\boldsymbol{X}}(x_1, x_2) = C(F_1(x_1), F_2(x_2)) \tag{1}$$

Conversely, if C is a copula and F_1 and F_2 are distributions of X_1 and X_2, respectively, then the function $F_{\boldsymbol{X}}(x_1, x_2)$ defined by (1) is a bivariate distribution function with margins F_1 and F_2.

- In the case of continuous random variables, the copula is unique.

- C couples the marginal distributions, entirely describing the dependence structure between them, separately from the margins themselves.

Introduction to Copulas

Example: Pareto's copula

- Consider the bivariate Pareto previously discussed

$$F_{X_1,X_2}(x_1,x_2) = F_1(x_1) + F_2(x_2) - 1 + \left[(1 - F_1(x_1))^{-1/\alpha} + (1 - F_2(x_2))^{-1/\alpha} - 1\right]^{-\alpha}$$

- We see that the underlying copula function is

$$C(u_1,u_2) = u_1 + u_2 - 1 + \left[(1-u_1)^{-1/\alpha} + (1-u_2)^{-1/\alpha} - 1\right]^{-\alpha}, \quad (u_1,u_2) \in [0,1]^2$$

- $\alpha > 0$ is the dependence parameter
- The copula construction does not constrain the choice of the marginal models!

Example: Clayton's copula

$$C(u_1,u_2) = \left[u_1^{-\alpha} + u_2^{-\alpha} - 1\right]^{-1/\alpha}, \quad (u_1,u_2) \in [0,1]^2$$

- $\alpha > 0$ is the dependence parameter

Sklar's theorem and copulas

Partial derivarives

The partial derivatives of C exist almost everywhere, the functions

$$u_1 \mapsto \frac{\partial}{\partial u_2} C(u_1, u_2) \quad \text{and} \quad u_2 \mapsto \frac{\partial}{\partial u_1} C(u_1, u_2)$$

being defined and non-decreasing almost everywhere and verifying, for $i = 1, 2$

$$0 \leqslant \frac{\partial}{\partial u_i} C(u_1, u_2) \leqslant 1, \qquad u_j \in [0, 1], \ \ j \neq i$$

Conditional probabilities

The copula partial derivatives are strictly related with the conditional probabilities of X_1 and X_2:

- Given a random vector (X_1, X_2) with joint distribution given by copula C then we have

$$\begin{aligned} P(X_2 \leqslant x_2 | X_1 = x_1) &= \frac{\partial}{\partial u_1} C(u_1, u_2) \Big|_{(u_1,u_2)=(F_1(x_1),F_2(x_2))} \\ P(X_1 \leqslant x_1 | X_2 = x_2) &= \frac{\partial}{\partial u_2} C(u_1, u_2) \Big|_{(u_1,u_2)=(F_1(x_1),F_2(x_2))} \end{aligned}$$

Sklar's theorem and copulas

Joint density function

Let X_1 and X_2 be continuous.
To obtain the joint density function, we resort to the second order crossed partial derivate of the copula, which exists almost everywhere, and is denoted the copula density c:

$$c(u_1, u_2) = \frac{\partial^2}{\partial u_1 \partial u_2} C(u_1, u_2), \quad (u_1, u_2) \in [0,1] \times [0,1]$$

If the marginal distributions F_1 and F_2 are continuous functions with respective marginal densities f_1 and f_2, then the joint density function of (X_1, X_2) is given by

$$f_{\boldsymbol{X}}(x_1, x_2) = \underbrace{f_1(x_1) f_2(x_2)}_{\substack{\textbf{independence} \\ \textbf{joint pdf}}} \times \underbrace{c\left(F_1(x_1), F_2(x_2)\right)}_{\textbf{dependence structure}}$$

Remark

- It is evident that the joint density function can be decoupled in two parts, the part corresponding to independence and the part enclosing the dependence structure.
- The dependence structure is fully described by the copula density, which is, for that reason, also known as *dependence function*.

Sklar's theorem and copulas

Invariance under increasing transformations

Let C be a copula associating the random variables (X_1, X_2) and h_1 and h_2 non-decreasing functions. Then the random vector $(h_1(X_1), h_2(X_2))$ also possesses copula C.

Remark

- It is due to this property that copula-based dependence measures, such as the Kendall's and Spearman's rank correlations, are invariant to strictly increasing functions.
- Because the copula links the ranks of random variables, transformations that preserve the ranks of random variables will also preserve the copula
- For example, regarding the multivariate distribution linking r.v., it makes no difference whether one models the random variables or their logarithms. The resulting copulas for the multivariate distributions are identical.

Sklar's theorem and copulas

Survival copula

The survival copula associated to a copula C is

$$\overline{C}(u_1, u_2) = u_1 + u_2 - 1 + C(1 - u_1, 1 - u_2) \tag{2}$$

Remarks

- The survival copula is a copula itself, fulfilling all conditions in definition of copula when evaluated at $(1 - u_1, 1 - u_2) \in [0, 1] \times [0, 1]$.
- We have that
$$\overline{C}(1 - u_1, 1 - u_2) = C(u_1, u_2) + 1 - u_1 - u_2$$
which results in
$$\overline{C}(1 - u_1, 1 - u_2) = P(X_1 > x_1, X_2 > x_2)$$
i.e. $S_{\boldsymbol{X}}(x_1, x_2) = P(X_1 > x_1, X_2 > x2) = \overline{C}(S_1(x_1), S_2(x_2))$
- Thus, survival copulas can be used to express the joint survival probability function.
- Notice that $P(X_1 > x_1, X_2 > x2) = \overline{C}(S_1(x_1), S_2(x_2)) \neq \overline{C}(F_1(x_1), F_2(x_2))$

Sklar's theorem and copulas

Example: survival copula of Pareto's copula

- The survival copula of Pareto's copula is

$$\overline{C}(u_1, u_2) = \left[u_1^{-1/\alpha} + u_2^{-1/\alpha} - 1\right]^{-\alpha}$$

- This is Clayton's copula with dependence parameter $1/\alpha$

Measures of dependence

Dependence measures

There are many ways of describing dependence or association between random variables

- Linear correlations
- Kendall's tau, τ_K
- Spearman's rho, ρ_S

Dependence

- In elliptical distribution context linear correlation is a natural summary of dependence
- In the non-elliptical distribution context intuition about correlation breaks and deeper understanding of dependence is needed to model risks
- Using copulas, measures of non-linear dependence can be explored, e.g. the Spearman's or Kendall's rank correlations.
- These dependence measures are copula-based (not moment based)
- Linear-correlation is not copula-based and can often be misleading when analysing dependencies.

Measures of dependence

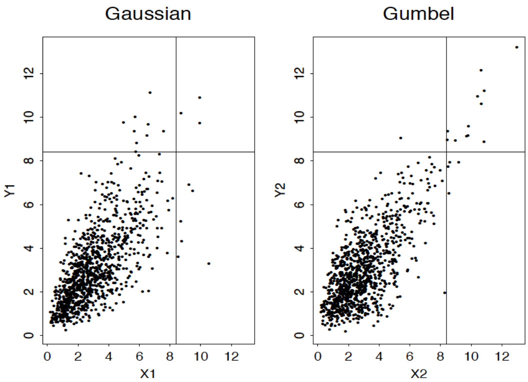


FIGURE 1. 1000 random variates from two distributions with identical Gamma(3,1) marginal distributions and identical correlation $\rho = 0.7$, but *different* dependence structures.

Image from: Embrechts, Macneil and Straumann, *Correlation and dependence in risk management: properties and pitfalls, 1999*

Measures of dependency

Three important copulas: Fréchet-Hoeffding bounds

Independence copula, where the dependence structure is non-existent:

$$C_I(u_1, u_2) = u_1 u_2, \quad (u_1, u_2) \in [0, 1] \times [0, 1]$$

Fréchet upper bound copula, which bounds all copulas, from above:

$$C_u(u_1, u_2) = \min(u_1, u_2), \quad (u_1, u_2) \in [0, 1] \times [0, 1]$$

also known as co-monotonic or minimum copula;
captures the relationship between two r.v. whose values are directly dependent on each other.

Fréchet lower bound copula, which bounds all copulas, from below:

$$C_l(u_1, u_2) = \max(0, u_1 + u_2 - 1), \quad (u_1, u_2) \in [0, 1] \times [0, 1]$$

also known as counter-monotonic or maximum copula;
captures the corresponding inverse relationship.

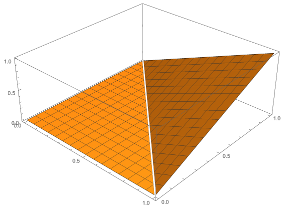
Measures of dependency

Remarks

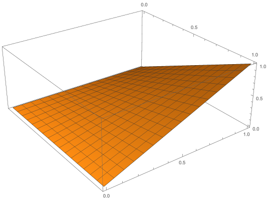
- Thus, given a copula C, the following inequalities always hold:

$$C_l(u_1, u_2) \leqslant C(u_1, u_2) \leqslant C_u(u_1, u_2), \quad (u_1, u_2) \in [0, 1] \times [0, 1]$$

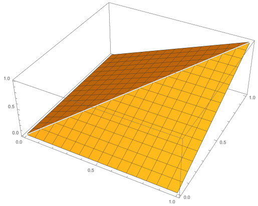
- The Fréchet-Hoeffding lower bound $C_l^d(\boldsymbol{u}) = \max(u_1 + \ldots + u_d - d + 1, 0)$ is not a copula for $d \geqslant 3$
- This is beacuse it is not possible to have three or more variables where each pair has a direct inverse relationship



$C_l(u_1, u_2) = \max(u_1 + u_2 - 1, 0)$



$C(u_1, u_2) = u_1 u_2$



$C_u(u_1, u_2) = \min(u_1, u_2)$

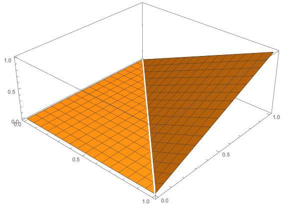
Measures of dependency

Measures of dependency

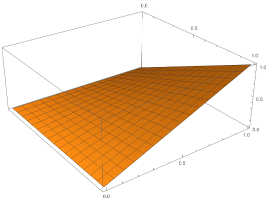
- In the bivariate case, C_l and C_u are themselves copulas, since if $U \sim Unif(0,1)$:

$$C_l(u_1, u_2) = P(U \leqslant u_1, 1 - U \leqslant u_2) \quad \text{and} \quad C_u(u_1, u_2) = P(U \leqslant u_1, U \leqslant u_2)$$

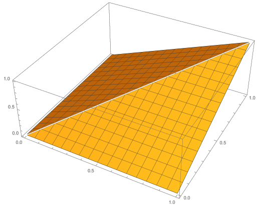
  so that C_l and C_u are the bivariate distribution functions of vectors $(U, 1-U)$ and (U, U), respectively.
  - C_l: the cdf of $(U, 1-U)$ has its mass on the diagonal between $(0,1)$ and $(1,0)$
  - C_u: the cdf of (U, U) has its mass on the diagonal between $(0,0)$ and $(1,1)$
- In these cases we say that C_l and C_u describe perfect negative and perfect positive dependence, respectively



$C_l(u_1, u_2) = \max(u_1 + u_2 - 1, 0)$



$C(u_1, u_2) = u_1 u_2$



$C_u(u_1, u_2) = \min(u_1, u_2)$

Measures of dependency

Theorem

Let (X_1, X_2) have one of the follwoing copulas:

$$C_l(F_1(x_1), F_2(x_2)) = \max(F_1(x_1) + F_2(x_2) - 1, 0) \quad \text{or} \quad C_u(F_1(x_1), F_2(x_2)) = \min(F_1(x_1), F_2(x_2))$$

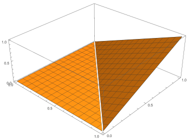
Then there exist two monotonic functions $u, v : \mathbb{R} \to \mathbb{R}$ and a real-valued r.v. Z such that

$$(X_1, X_2) =_d (u(Z), v(Z))$$

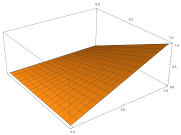
with

- u increasing and v decreasing in case of C_l
- u and v increasing in case of C_u

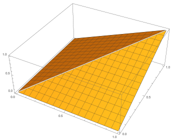
The converse of this result is also true.



$C_l(u_1, u_2) = \max(u_1 + u_2 - 1, 0)$



$C(u_1, u_2) = u_1 u_2$

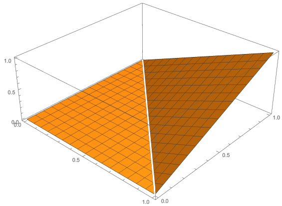


$C_u(u_1, u_2) = \min(u_1, u_2)$

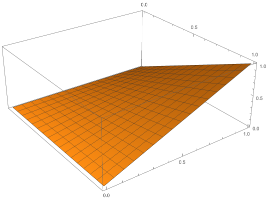
Measures of dependency

Definition

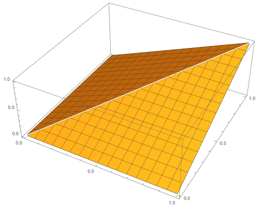
- If (X_1, X_2) has the copula C_l, then X_1 and X_2 are said to be **countermonotonic**
- If (X_1, X_2) has the copula C_u, then X_1 and X_2 are said to be **comonotonic**



$C_l(u_1, u_2) = \max(u_1 + u_2 - 1, 0)$



$C(u_1, u_2) = u_1 u_2$



$C_u(u_1, u_2) = \min(u_1, u_2)$

Measures of dependency

Concordance

Let (x_1, x_2) and $(\tilde{x}_1, \tilde{x}_2)$ be two observations from a vector (X_1, X_2).
Then (x_1, x_2) and $(\tilde{x}_1, \tilde{x}_2)$ are said to be

- **concordant** if $(x_1 - \tilde{x}_1)(x_2 - \tilde{x}_2) > 0$
  (if the line segment connecting (x_1, x_2) and $(\tilde{x}_1, \tilde{x}_2)$ has positive slope)
- **discordant** if $(x_1 - \tilde{x}_1)(x_2 - \tilde{x}_2) < 0$
  (if the line segment connecting (x_1, x_2) and $(\tilde{x}_1, \tilde{x}_2)$ has negative slope)

Intuitive idea:

- Two r.v.'s X_1 and X_2 are concordant when large values of X_1 go together with large values of X_2.

Measures of dependency

Probabilities of concordance and discordance

Let (X_1, X_2) and $(\tilde{X}_1, \tilde{X}_2)$ be two independent pairs of r.v.'s with the same margins, F_1 for X_1 and $\tilde{X}_1$, and F_2 for X_2 and $\tilde{X}_2$. Then

- $P((X_1 - \tilde{X}_1)(X_2 - \tilde{X}_2) > 0)$ is the probability of concordance
- $P((X_1 - \tilde{X}_1)(X_2 - \tilde{X}_2) < 0)$ is the probability of discordance

Difference between the probabilities of concordance and discordance

If, furthermore, the joint distribution of (X_1, X_2) is given by copula C and the joint distribution of $(\tilde{X}_1, \tilde{X}_2)$ is given by copula $\tilde{C}$, and X_1 and X_2 are continuous r.v.'s, then

$$P((X_1 - \tilde{X}_1)(X_2 - \tilde{X}_2) > 0) - P((X_1 - \tilde{X}_1)(X_2 - \tilde{X}_2) < 0) = 4 \int_0^1 \int_0^1 \tilde{C}(u_1, u_2) dC(u_1, u_2) - 1$$

Where

$$\begin{aligned} dC(u_1, u_2) &= \overbrace{\frac{\partial^2 C(u_1, u_2)}{\partial u_1 \partial u_2}}^{c(u_1, u_2)} f_1(F^{-1}(u_1)) f_2(F^{-1}(u_2)) du_1 du_2 \\ &= c(F_1(x_1), F_1(x_2)) f_1(x_1) f_2(x_2) dx_1 dx_2 = f_{\mathbf{X}}(x_1, x_2) d_{x_1} d_{x_2} \end{aligned}$$

Measures of dependency

Desired properties of depedence measures: concordance measures (Denuit et.al. 2005)

The function $r(\cdot,\cdot)$ assigning a real number to any (bivariate cdf of a) pair of real valued r.v.'s (X_1, X_2) is a **concordance measure** if it fulfills the following properties:

- P1 $r(X_1, X_2) = r(X_2, X_1)$ (symmetry)
- P2 $-1 \leqslant r(X_1, X_2) \leqslant 1$ (normalization)
- P3 $r(X_1, X_2) = 1$ if and only if X_1 and X_2 are comontonic
- P4 $r(X_1, X_2) = -1$ if and only if X_1 and X_2 are countermontonic
- P5 $t : \mathbb{R} \to \mathbb{R}$ strictly monotonic, then
$$r(t(X_1), X_2) = \begin{cases} r(X_1, X_2), & \text{if } t \text{ is increasing} \\ -r(X_1, X_2), & \text{if } t \text{ is decreasing} \end{cases}$$

Remarks

- Linear correlation fulfills properties $P1$ and $P2$ only
- One might think of other desirable properties. These are however incompatible with P1-P5
- E.g., another interesting property could be
$$r(X_1, X_2) = 0 \iff X_1 \text{ independent from } X_2$$
Unfortunately, this contradicts $P5$.

Measures of dependency

Kendall's tau τ_K rank correlation coefficient

- Difference between the probabilities of concordance and discordance:

$$\tau_K = P((X_1 - \tilde{X}_1)(X_2 - \tilde{X}_2) > 0) - P((X_1 - \tilde{X}_1)(X_2 - \tilde{X}_2) < 0)$$

  For (X_1, X_2) and $(\tilde{X}_1, \tilde{X}_2)$, independent and identically distributed continuous bivariate r.v.'s with marginals F_1 for X_1 and $\tilde{X}_1$, and F_2 for X_2 and $\tilde{X}_2$

- It is easy to sea that

$$\tau_K = E[sign(X_1 - \tilde{X}_1)(X_2 - \tilde{X}_2)]$$

- If the vector of continuous r.v.'s (X_1, X_2) has copula C, then

$$\tau_K = 4\int_0^1\int_0^1 C(u_1, u_2)\, dC(u_1, u_2) - 1 = 4E[C(U, V)] - 1$$

Measures of dependency

Spearman's rho ρ_S rank correlation coefficient

- Proportional to the difference between the probabilities of concordance and discordance between vectors (X_1, X_2) and $(\tilde{X}_1, X_2^*)$, where (X_1, X_2), $(\tilde{X}_1, \tilde{X}_2)$ and (X_1^*, X_2^*) are independent copies:

$$\tau_K = 3\left(P((X_1 - \tilde{X}_1)(X_2 - X_2^*) > 0) - P((X_1 - \tilde{X}_1)(X_2 - X_2^*) < 0)\right)$$

- Note that $\tilde{X}_1$ and X_2^* are independent
- It is proportional to the probability of concordance minus the probability of discordance for a pair of random vectors with the same marginals, where one of the pairs has independent components
- If the vector of continuous r.v.'s (X_1, X_2) has copula C, then

$$\rho_S = 12 \int_0^1 \int_0^1 C(u_1, u_2)\, du_1 du_2 - 3 = 12 \int_0^1 \int_0^1 (C(u_1, u_2) - u_1 u_2)\, du_1 du_2$$

- We have that

$$\rho_S(X_1, X_2) = \rho(F_1(X_1), F_2(X_2)) = \frac{Cov(F_1(X_1), F_2(X_2))}{\sqrt{Var(F_1(X_1))}\sqrt{Var(F_2(X_2))}}$$

Measures of dependency

Properties of Kendall's and Spearman's rank correlations: concordance measures

Let $r_C(X_1, X_2) = \tau_K$ or $r_C(X_1, X_2) = \rho_S$, for copula C

1. They are defined for every pair (X_1, X_2) of continuous r.v.'s
2. $-1 \leqslant r_C(X_1, X_2) \leqslant 1$, $r_C(X_1, X_1) = 1$ and $r_C(X_1, -X_1) = -1$
3. $r_C(X_1, X_2) = r_C(X_2, X_1)$
4. X_1 and X_2 independent, then $r_C(X_1, X_2) = 0$
5. $r_C(-X_1, X_2) = r_C(X_1, -X_2) = -r_C(X_1, X_2)$
6. If C and $\tilde{C}$ are copulas such that $C(u_1, u_2) \leqslant \tilde{C}(u_1, u_2)$ and $\overline{C}(1-u_1, 1-u_2) \leqslant \overline{\tilde{C}}(1-u_1, 1-u_2)$, for all $(u_1, u_2) \in [0,1]^2$ (C is smaller than $\tilde{C}$, $C \prec \tilde{C}$), then
$$r_C(X_1, X_2) \leqslant r_{\tilde{C}}(X_1, X_2)$$
7. If (X_1^n, X_2^n) is a sequence of continuous r.v. with copulas C_n and if C_n converges pointwise to C, then
$$\lim_{n \to \infty} r_{C_n}(X_1, X_2) = r_C(X_1, X_2)$$

Furthermore

- $r_C(X_1, X_2) = 1 \iff C = C_u$
- $r_C(X_1, X_2) = -1 \iff C = C_l$

Measures of dependency

Remarks

- Kendall's tau and Spearman's rho are rank correlations in that, when the marginals are continuous, they depend only on the bivariate copula and not on the marginals.
- Kendall's tau and Spearman's rho for the r.v. (X_1, X_2) are invariant under strictly increasing componentwise transformations. This is not true for Pearson's linear correlation.
- The linear correlation coefficient, based on the covariance of two r.v.'s, is not preserved by copulas: two pairs of correlated variables with the same copula can have different correlations.
- Kendall's correlation is constant for the copula: any correlated variates with the same copula will have the τ_K of that copula

Tail dependency

Tail dependency

- Amount of dependence in the upper-right-quadrant tail, or lower-left-quadrant tail, of a bivariate distribution
- Relevant for the study of dependence between extreme outcomes
- (from Loss Models) “although in 'normal times' there may be little correlation, in 'bad times' there may be signifficant correlation between risks. ('Everything seems to go wrong at once.')”

Tail dependency

- Measures of tail dependence have been developed to evaluate how strong the correlation is in the upper (or lower) tails.
- It turns out that tail dependence between two continuous r.v.'s X_1 and X_2 is a copula property
- Hence, the amount of tail dependence is invariant under strictly increasing transformations of X_1 and X_2

Tail dependency

Index of upper tail dependence

Consider two continuous r.v.'s X_1 and X_2 with marginal distributions $F_1(x_1)$ and $F_2(x_2)$.

- The index of upper tail dependence, λ_U, is defined as

$$\lambda_U = \lim_{u \to 1} P(X_1 > F_1^{-1}(u) | X_2 > F_2^{-1}(u))$$

  provided the limit $\lambda_U \in [0, 1]$ exists.
- Obviously $0 \leqslant \lambda_U \leqslant 1$
- If $\lambda_U \in (0, 1]$, X_1 and X_2 are said to be **asymptotically dependent** (in the upper tail)
- If $\lambda_U = 0$, they are **asymptotically independent** (upper tail independent)

Tail dependency

Remarks

- The index of upper tail dependence measures the chances that X_1 is very large if it is known that X_2 is very large, where "very large" is measured in terms of equivalent quantiles.
- If (X_1, X_2) has joint distribution given by copula C, then

$$\lambda_U = \lim_{u \to 1} \frac{1 - 2u + C(u,u)}{1-u} = \lim_{u \to 1} \frac{\overline{C}(1-u, 1-u)}{1-u}$$

  i.e., tail dependence can be measured by looking at the copula rather than the original distribution.

Tail dependency

Index of lower tail dependence

Consider two continuous r.v.'s X_1 and X_2 with marginal distributions $F_1(x_1)$ and $F_2(x_2)$.

- The index of lower tail dependence, λ_L, is defined as

$$\lambda_L = \lim_{u \to 0} P(X_1 \leqslant F_1^{-1}(u) | X_2 \leqslant F_2^{-1}(u))$$

- If (X_1, X_2) has joint distribution given by copula C, then

$$\lambda_L = \lim_{u \to 0} \frac{C(u,u)}{u}$$

Tail dependency

Index of tail dependence

- We will focus in the upper tail dependence (upper-right-quadrant tail)
- The index of tail dependence is a very useful measure in describing and comparing copulas.

Examples

- The tail indices of the independence copula are $\lambda_U = \lambda_L = 0$ (tail independence)
- For the comonotonic copula $C_u(u_1, u_2) = \min(u_1, u_2)$ we have that $\lambda_U = \lambda_L = 1$ (perfet tail dependence)

Tail dependency

Example: Pareto's copula

$$\lambda_U = \lim_{u\to 1} \frac{\left[2(1-u)^{-1/\alpha} - 1\right]^{-\alpha}}{1-u} = 2^{-\alpha}$$

and

$$\lambda_L = \lim_{u\to 0} \frac{2u - 1 + \left[2(1-u)^{-1/\alpha} - 1\right]^{-\alpha}}{u} = 0$$

Pareto's copula has upper tail dependence and no lower tail dependence.

Example: Clayton's copula

$$\lambda_U = \lim_{u\to 1} \frac{1 - 2u + \left(2u^{-\alpha} - 1\right)^{-1/\alpha}}{1-u} = 0$$

and

$$\lambda_L = \lim_{u\to 0} \frac{\left(2u^{-\alpha} - 1\right)^{-1/\alpha}}{u} = 2^{-1/\alpha}$$

Clayton's copula (for $\alpha > 0$) has lower tail dependence and no upper tail dependence.
(Recall that Clayton's copula is Pareto's survival copula with dependence parameter $1/\alpha$)

Archimedean copulas

The generator function

$\phi : [0,1] \to \mathbb{R}^+$, such that

- possibly infinite, with continuous first and second derivatives on $(0,1)$
- $\phi(1) = 0$
- $\phi'(t) < 0$ (strictly decreasing), for all $t \in [0,1]$
- $\phi''(t) > 0$ (convex), for all $t \in [0,1]$
- the inverse generator $\phi^{-1}(t)$ is completely monotonic on $[0,\infty]$:

$$(-1)^n \frac{d^n}{dt^n}\phi^{-1}(t) \geqslant 0, \qquad n = 1,2,3\dots$$

Archimedean copula

$$C(u_1,u_2) = \phi^{-1}\left[\phi(u_1) + \phi(u_2)\right]$$

- The support is the area in the unit square where $\phi(u_1) + \phi(u_2) \leqslant \phi(0)$
- If $\phi(0) = \infty$ (ϕ is a strict generator), then the support is the entire unit square and the copula is said to be a strict Archimedean copula

Archimedean copulas

Remark

The class of Archimedean copulas allow for a great variety of different dependence structures

Kendall's tau for Archimedean copulas (see e.g. Nelsen, 2013)

$$\tau_K = 1 + 4\int_0^1 \frac{\phi(t)}{\phi'(t)}\,dt$$

This formula allows very easy comparisons of Archimedean copulas based solely on their generators.

Archimedean copulas

Upper tail dependence for Archimedean copulas

$$\lambda_U = 2 - 2 \lim_{t \to 0} \frac{\frac{d}{dt}\phi^{-1}(2t)}{\frac{d}{dt}\phi^{-1}(t)}$$

- provided that $\lim_{t \to 0} \frac{d}{dt}\phi^{-1}(t) = -\infty$
- If $\lim_{t \to 0} \frac{d}{dt}\phi^{-1}(t) \neq -\infty$, there is no upper tail dependence

Lower tail dependence for Archimedean copulas

$$\lambda_L = 2 \lim_{t \to +\infty} \frac{\frac{d}{dt}\phi^{-1}(2t)}{\frac{d}{dt}\phi^{-1}(t)}$$

- provided that $\lim_{t \to +\infty} \frac{d}{dt}\phi^{-1}(t) = 0$
- If $\lim_{t \to +\infty} \frac{d}{dt}\phi^{-1}(t) \neq 0$, there is no lower tail dependence

Archimedean copulas

Independence copula

$$\phi(u) = -\ln u$$
$$C(u_1, u_2) = u_1 u_2$$
$$\tau_K = 0$$
$$\lambda_U = \lambda_L = 0$$

Countermonotonic copula (Fréchet-Hoeffding lower bound)

$$\phi(u) = 1 - u$$
$$C(u_1, u_2) = \max(u_1 + u_2 - 1, 0)$$
$$\tau_K = -1$$

Archimedean copulas

Clayton copula

$$\phi(u) = \frac{t^{-\alpha} - 1}{\alpha}, \qquad \alpha \in [-1, +\infty) \setminus \{0\}$$

$$C_\alpha(u_1, u_2) = \max((u_1^{-\alpha} + u_2^{-\alpha} - 1)^{-1/\alpha}, 0)$$

$$C_\alpha(u_1, u_2) = (u_1^{-\alpha} + u_2^{-\alpha} - 1)^{-1/\alpha}, \qquad \text{for } \alpha > 0 \quad \text{(sctrict Archimedean copula)}$$

$$\tau_K = \frac{\alpha}{\alpha + 2}$$

$$\lambda_U = 0 \quad \text{(no upper tail dependence)}$$

$$\lambda_L = 2^{-1/\alpha} \quad \text{(lower tail dependence for } \alpha > 0\text{)}$$

$$\lim_{\alpha \to -1} C(u_1, u_2) = C_l(u_1, u_2) = \max(u_1 + u_2 - 1, 0)$$

$$\lim_{\alpha \to 0} C(u_1, u_2) = u_1 u_2$$

$$\lim_{\alpha \to +\infty} C(u_1, u_2) = C_u(u_1, u_2) = \min(u_1, u_2)$$

Archimedean copulas

Frank copula

$$\phi(u) = -\ln\left[\frac{e^{-\alpha u}-1}{e^{-\alpha}-1}\right], \qquad \alpha \in (-\infty,+\infty)\setminus\{0\}$$

$$C_\alpha(u_1,u_2) = -\frac{1}{\alpha}\log\left(1+\frac{(e^{-\alpha\, u_1}-1)(e^{-\alpha\, u_2}-1)}{e^{-\alpha}-1}\right) \quad \text{(sctrict Archimedean copula)}$$

$$\tau_K = 1-4\frac{1-D_1(\alpha)}{\alpha} \quad \text{where } D_k(x) \text{ is the Debye function:} \quad D_k(x) = \frac{k}{x^k}\int_0^x \frac{t^k}{e^t-1}dt$$

$$\lambda_U = 0 \quad \text{(no upper tail dependence)}$$

no lower tail dependence

$$\lim_{\alpha\to-\infty} C(u_1,u_2) = C_l(u_1,u_2) = \max(u_1+u_2-1,0)$$

$$\lim_{\alpha\to 0} C(u_1,u_2) = u_1 u_2$$

$$\lim_{\alpha\to+\infty} C(u_1,u_2) = C_u(u_1,u_2) = \min(u_1,u_2)$$

Archimedean copulas

Gumbel copula (Gumbel-Hougaard)

$$\phi(u) = (-\ln u)^{\alpha}, \qquad \alpha \geqslant 1$$

$$C_{\alpha}(u_1, u_2) = \exp\left(-\left[(-\ln u_1)^{\alpha} + (-\ln u_2)^{\alpha}\right]^{1/\alpha}\right) \quad \text{(sctrict Archimedean copula)}$$

$$\tau_K = 1 - \frac{1}{\alpha}$$

$$\lambda_U = 2 - 2^{1/\alpha} \quad \text{(upper tail dependence for } \alpha \neq 1\text{)}$$

no lower tail dependence

$$\lim_{\alpha \to 1} C(u_1, u_2) = u_1 u_2$$

$$\lim_{\alpha \to +\infty} C(u_1, u_2) = C_u(u_1, u_2) = \min(u_1, u_2)$$

Archimedean copulas

Joe copula

$$\phi(u) = -\ln\left[1-(1-u)^{\alpha}\right], \qquad \alpha \geqslant 1$$

$$C_{\alpha}(u_1,u_2) = 1-\left[(1-u_1)^{\alpha}+(1-u_2)^{\alpha}-(1-u_1)^{\alpha}(1-u_2)^{\alpha}\right]^{1/\alpha} \quad \text{(sctrict Arch. copula)}$$

τ_K has no (convenient) closed form

$$\lambda_U = 2-2^{1/\alpha} \quad \text{(upper tail dependence for } \alpha \neq 1\text{)}$$

no lower tail dependence

Archimedean copulas

BB1 copula

$$\phi(u) = (u^{-\delta} - 1)^{\alpha}, \qquad \delta > 0, \alpha \geqslant 1$$

$$C_\alpha(u_1, u_2) = \left[1 + \left((u_1^{-\delta} - 1)^{\alpha} + (u_2^{-\delta} - 1)^{\alpha}\right)^{1/\alpha}\right]^{-1/\delta}$$

$$\lambda_U = 2 - 2^{1/\alpha} \quad \text{(upper tail dependence for } \alpha \neq 1\text{)}$$

has lower tail dependence

BB3 copula

$$\phi(u) = e^{\delta(-\ln u)^{\alpha}} - 1, \qquad \delta > 0, \alpha \geqslant 1$$

$$C_\alpha(u_1, u_2) = \exp\left[-\frac{1}{\delta}\left(\ln\left(e^{\delta(-\ln u_1)^{\alpha}} + e^{\delta(-\ln u_2)^{\alpha}} - 1\right)\right)\right]^{1/\alpha}$$

$$\lambda_U = 2 - 2^{1/\alpha} \quad \text{(upper tail dependence for } \alpha \neq 1\text{)}$$

has lower tail dependence

Elliptical copulas

Elliptical copulas

- Copulas of elliptical distributions
- The two main models are the Gaussian copula, associated with the multivariate normal distribution, and the student t copula, associated with the multivariate t distribution

Elliptical copulas

Let $F_{\boldsymbol{X}}$ be the multivariate cdf of an elliptical distribution and let F_1 and F_2 be the marginal cdf's. The elliptical copula determined by $F_{\boldsymbol{X}}$ is

$$C(u_1, u_2) = F_{\boldsymbol{X}}(F_1^{-1}(u_1), F_2^{-1}(u_2))$$

- The extension to d dimensions is obvious

Gaussian copula

Gaussian copula

$$C(u_1, u_2) = \mathbf{\Phi}_{\boldsymbol{P}}(\Phi^{-1}(u_1), \Phi^{1}(u_2))$$

where $\phi(x)$ is the standard univariate normal cdf and $\mathbf{\Phi}_{\boldsymbol{P}}$ is the joint cdf of the standard multivariate normal r.v., with mean 0 and variance 1 in each component and correlation mattrix $\boldsymbol{P}$

- The extension to d dimensions is obvious
- Because the correlation matrix contains $d/(d-1)/2$ pairwise correlations, this is the number of parameters in the copula
- There is no simple closed form for the copula

$$C(u_1, u_2) = \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)}} \, dx_1 \, dx_2$$

Gaussian copula

Remarks

- If all correlations in $\boldsymbol{P}$ are zero, the Gaussian copula reduces to the independence copula
- It is easy to simulate observations from this copula

Gaussian copula

$$\tau_K = \frac{2}{\pi}\arcsin(\rho)$$
$$\rho_S = \frac{6}{\pi}\arcsin\frac{\rho}{2}$$

$$\begin{cases} \lambda_U = \lambda_L = 0, & \text{if } \rho \neq 1 \\ \lambda_U = \lambda_L = 1, & \text{if } \rho = 1 \end{cases}$$

Student t copula

Student t copula

$$C(u_1, u_2) = \boldsymbol{t}_{\nu,\boldsymbol{P}}(t_\nu^{-1}(u_1), t_\nu^{1}(u_2))$$

where $t_\nu(x)$ is the standard univariate t cdf with ν degrees of freedom and $\boldsymbol{t}_{\nu,\boldsymbol{P}}$ is the joint cdf of the standard multivariate t r.v. with ν degrees of freedom, with correlation mattrix $\boldsymbol{P}$

- The extension to d dimensions is obvious
- There is no simple closed form for the copula

$$C(u_1, u_2) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \int_{-\infty}^{t_\nu^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} \left[1 + \frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{\nu(1-\rho^2)}\right]^{-1-\frac{\nu}{2}} dx_1\, dx_2$$

Student t copula

Student t copula

$$\tau_K = \frac{2}{\pi}\arcsin(\rho)$$

$$\lambda_U = 2t_{\nu+1}\left(-\sqrt{\frac{1-\rho}{1+\rho}(\nu+1)}\right)$$

has lower tail dependence

Remarks

- If all correlations in $\boldsymbol{P}$ are zero, the t copula **does not** reduces to the independence copula
- $\rho = 0 \not\Rightarrow \lambda_U = 0$
- For fixed correlation coefficient ρ, the degree of upper tail dependence can be tuned tgrough the single parameter ν

Elliptical copulas

Remarks (Embrechts et.al., 2002)

- The Gaussian copula has the property of asymptotic independence. Regardless of how high a correlation we choose, if we go far enough into the tail, extreme events appear to occur independently in each margin.
- In contrast, the t-copula displays asymptotic upper tail dependence even for negative and zero correlations, with dependence rising as the degrees of freedom parameter decreases and the marginal distributions become heavy-tailed

Extreme value copulas

Extreme value copulas

- Important class of copulas associated with the extreme value distributions
- Defined in terms of the scaling property of extreme value distributions

Extreme value copulas

An extreme value (EV) copula is given by

$$C(u_1^n, \dots, u_d^n) = C^n(u_1, \dots, u_d), \quad \forall (u_1, \dots, u_d) \in [0,1]^d, \quad n > 0$$

Extreme value copulas

- The scale property results in the EV copula having the **stability of the maximum (or max-stable)** property:
$$C_{\max}(u_1^n, u_2^n) = C^n(u_1, u_2)$$
If the copula is an EV copula, than the copula for the maxima is also an EV copula
- EV copulas are those with max-stable property: the copula associated with (M_{X_1}, M_{X_2}) is also C

Extreme value copulas

Extreme value copulas

In the bivariate case, EV copulas can be represented as

$$C(u_1, u_2) = \exp\left[\ln(u_1 u_2) A\left(\frac{\ln u_1}{\ln(u_1 u_2)}\right)\right]$$

where A is a dependence function

$$A(w) = \int_0^1 \max(x(1-w), w(1-x))dH(x), \qquad \forall w \in [0,1]$$

and H is a distribution function.
$A(w)$ must be convex verifying

$$\max(w, 1-w) \leqslant A(w) \leqslant 1, \qquad 0 < w < 1$$

Extreme value copulas

- $A(w) = 1$ leads to the independence copula
- $A(w) = \max(w, 1-w)$ leads to perfect correlation, i.e., perfect dependency with $C(u,u) = u$

Extreme value copulas

Extreme value copulas

- $\lambda_U = 2 - 2A(1/2)$
- There are several well-known copulas in this class

Gumbel copula

$$A(w) = [w^{\alpha} + (1-w)^{\alpha}]^{1/\alpha}, \qquad \alpha \geqslant 0$$

- $\lambda_U = 2 - 2A(1/2) = 2 - 2^{1/\alpha}$

Galambos copula

$$A(w) = 1 - \left[w^{-\alpha} + (1-w)^{-\alpha}\right]^{-1/\alpha}, \qquad \alpha > 0$$

- It is not an Archimedean copula

$$C(u_1, u_2) = u_1 u_2 \exp\left[\left((-\ln u_1)^{-\alpha} + (-\ln u_2)^{-\alpha}\right)^{-1/\alpha}\right]$$

- $\lambda_U = 2 - 2A(1/2) = 2^{-1/\alpha}$