Probability and Stochastic Processes

Master in Actuarial Sciences

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General Notions of Stochastic Processes

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Stochastic Process

Often the systems we consider evolve in time and we are interested in their dynamic behaviour, usually involving some randomness.
Examples are

- the length of a queue
- the temperature outside
- the number of students passing Statistics I at ISEG along time
- the number of claims in a portfolio along time

Some definitions

Definition: stochastic process

Given a probability space $(\Omega, \mathcal{F}, P)$ and an arbitrary set T, a **stochastic process** is a real and finite function $X(t,\omega)$ defined in $T \times \Omega$, and that for each fixed t $X(t,\omega)$ is a measurable function of $\omega \in \Omega$

$$\{X(t,\omega) : t \in T\}$$

Thus

- **A stochastic process $\{X_t\}_{t \in T}$ is a family of random variables $X_t : \Omega \to \mathbb{R}$ indexed by a parameter t (usually the time).**
- For each fixed t, X_t is a random variable

Common notation:

$$\{X(t)\,;t \in T\}\,, \{X(t)\}_{t \in T}\,, \{X_t\,;t \in T\}\,, \{X_t\}_{t \in T}\,.$$

$$\{X(t)\,;t \geq 0\}\,, \{X(t)\}_{t \geq 0}\,, \{X_t\,;t = 0, 1, \cdots\}\,, \{X_t\}_{t=0}^{\infty}\,.$$

Some definitions

Definition: parameter space

T is called the parameter space.

Definition: state space

The set pf all possible values of X_t is the state space S.

Definition: trajectory or sample path

Joint realization of the random variables X_t for all $t \in T$. It is a function from T to S.

Examples

- sample path of the number of claims
- sample path of the aggregate claim amount

Specification

Joint distribution function

For an arbitrary finite set $t_1, t_2, \ldots, t_n$ of values $t \in T$, the corresponding r.v.'s $X(t_1), X(t_2), \ldots, X(t_n)$ have joint distribution function

$$F(x_1, x_2, \ldots, x_n\,;\, t_1, t_2, \ldots, t_n) = P(X(t_1) \leqslant x_1, \ldots, X(t_n) \leqslant x_n) \tag{1}$$

Definition: temporal law

The temporal law $\{X(t); t \in T\}$ is a family of functions (1) for $n = 1, 2, \ldots$ and all possible values t_j, $j = 1, 2, \ldots, n$

Alternatively, a stochastic process can be specified by the joint characteristic function given by

$$\varphi_{X(t_1), X(t_2), \ldots, X(t_n)}(s_1, s_2, \ldots, s_n) = E\left[\exp\left(i \sum_{k=1}^{n} s_k\, X(t_k)\right)\right]$$

Classification

Classification of a stochastic process

The classification of a stochastic process can be based on:

- the parametric space T
- the state space S
- the dependence relations betwen the random variables $X(t)$

Definition

- If T is countable, we say $\{X(t)\}_{t\in T}$ is a **discrete-time** stochastic process
- if T is a continuum, we say $\{X(t)\}_{t\in T}$ is a **continuous-time** stochastic process

Definition

- If S is countable, we say $\{X(t)\}_{t\in T}$ is a **discrete** stochastic process
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Classification

Definition: independent increments

A continuous-time stochastic process $\{X(t) : t \in T\}$ is said to have **independent increments** if, for all $t_0 < t_1 < \cdots < t_n < t$, the random variables

$$X(t_1) - X(t_0),\ X(t_2) - X(t_1),\ \ldots,\ X(t_n) - X(t_{n-1})$$

are independent.

Independent increments

For a process with independent increments, the probability law of $X(t)$ and $X(t) - X(s)$, for all t and s, specifies the process.

Classification

Definition: stationary increments

The stochastic process $\{X(t) : t \in T\}$ has **stationary increments** if the random variable

$$X(t_2 + h) - X(t_1 + h)$$

has the same distribution of

$$X(t_2) - X(t_1)$$

for all t_1, t_2 and $h \leqslant 0$.

Independent and stationary increments

If a process has independent and stationary increments, the probability law of $X(t)$ specifies the process.

Classification

Definition: counting process

A counting process is a stochastic process, in discrete or continuous time, whose state space is

$$S = \{0, 1, 2, \ldots\}$$

with the property that $X(t)$ is a non-decreasing function of t.

Classification

Definition: random walk

Stochastic process such that

- the initial value is X_0
- the process is observed at $t = 1, 2, \ldots$
- at time $t = 1$ the process jumps to $X_1 = X_0 + Z_1$, where the size of the jump Z_1 is a r.v. with a given distribution
- at time $t = 2$ the process jumps to $X_2 = X_1 + Z_2$, with Z_2 independent of Z_1, but with the same distribution

After t jumps

$$X_t = X_0 + Z_1 + \cdots + Z_t = X_{t-1} + Z_t$$

where Z_t, $t = 1, 2, \ldots$ is a sequence pf i.i.d. random variables.

Definition: simple random walk

Random walk where the r.v.'s Z_t take only the values $-1, 0$ and 1 with probabilities p, $1 - p - q$ and q.

Definition: symmetric random walk

Random walk where the r.v.'s Z_t take only the values -1 and 1 with probabilities $1/2$.

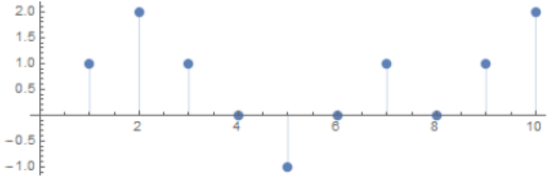
Classification

Definition: white noise

A white noise is a stochastic process that consists of a set of independent and identically distributed random variables. The random variables can be either discrete or continuous and the time set can be either discrete or continuous.

Example: white noise

$\{Z_t : t = 1, 2, \ldots\}$ in the random walk is a white noise.



Classification

Definition: stationary stochastic process

A stochastic process $\{X(t) : t \in T\}$ is said to be stationary or strictly stationary, if the joint distribution of

$$(X(t_1), X(t_2), \ldots, X(t_n))$$

and

$$(X(t_1 + h), X(t_2 + h), \ldots, X(t_n + h))$$

are identical for all h and for all $t_1, t_2, \ldots, t_n \in T$, and for all integer n.

Classification

Markov property

The future, given the present, does not depend on the past.

Definition: Markov process

Stochastic process satisfying the Markov property:

$$P(a < X(t_{n+1}) \leqslant b | X(t_1) = x_1, X(t_2) = x_2, \dots, X(t_n) = x_n) = P(a < X(t_{n+1}) \leqslant b | X(t_n) = x_n)$$

for all $t_1 < t_2 < \cdots < t_n < t_{n+1}$, $x_1, \dots, x_n$, a and b.
Or, in continuous time,

$$P(a < X(t+h) \leqslant b | \mathcal{F}_t) = P(a < X(t+h) \leqslant b | X_t)$$

where $\mathcal{F}_t$ is a natural filtration of the process (σ-algebra generated by the process up to time t).

Markov processes

- A stochastic process with independent increments is a Markov process.
- A random walk is a Markov process.

Classification

Definition: Poisson process

Counting process, with $X(0) = 0$, with independent and stationary increments and such that

$$P(X(t) = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}, \quad k = 0, 1, 2, \ldots$$

where $\lambda > 0$ is denoted the intensity of the process.

Poisson process

- The Poisson process is a Markov process.