Probability and Stochastic Processes

Master in Actuarial Sciences

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2018/2019

Continuous time inhomogeneous Markov chains

Continuous time homogeneous Markov chains

Definition: continuous time inhomogeneous Markov chain

Markov process, with countable state space S, in continuous time, such that there exists a probability function $p_{ij}(s,t)$ such that

$$p_{ij}(s,t) = P(X(t) = j | X(s) = i), \qquad \text{for all } 0 \leqslant s \leqslant t.$$

Remark

For each pair (s,t), with $0 \leqslant s \leqslant t$, the matrix

$$P(s,t) = \left[p_{ij}(s,t)\right]_{i,j \in S}$$

is stochastic, and $p_{ij}(t,t) = \delta_{ij}$, *i.e.* $P(t,t) = I$.

Chapman-Kolmogorov equations

Chapman-Kolmogorov equations

$$p_{ij}(s,t) = \sum_{k \in S} p_{ik}(s,u) p_{kj}(u,t), \qquad 0 \leqslant s \leqslant u \leqslant t.$$

Matrix of transition rates

- We will assume that the probability functions $p_{ij}(s,t)$ are differentiable.

Definition: transition rates

For all $i,j \in S$, the transition rate, transition intensity or force of transition from state i to state j is

$$q_{ij}(s) = \frac{\partial}{\partial t} p_{ij}(s,t)\Big|_{t=s}$$

Remarks

- Then, for all $t, h \geqslant 0$

$$\begin{aligned} P(X(s+h)=j|X(s)=i) &= p_{ij}(s,s+h) \\ &= p_{ij}(s,s) + q_{ij}(s)h + o(h) \qquad \text{as } h \to 0 \\ &= \delta_{ij} + q_{ij}(s)h + o(h) \qquad \text{as } h \to 0 \end{aligned}$$

- Hence $q_{ij}(s)$ is the (instantaneous) transition rate of the process from state i to state j
- The matrix

$$\mathbf{Q}(s) = \left[q_{ij}(s)\right]_{i,j\in S}$$

  is the **transition rate matrix**, or generator, of the process.

The forward differential equations

Theorem

$$\frac{\partial}{\partial t} p_{ij}(s,t) = \sum_{k \in S} p_{ik}(s,t) q_{kj}(t)$$

under the initial conditions $p_{ij}(s,s) = \delta_{ij}$.
In matrix form

$$\begin{cases} \frac{\partial}{\partial t}\mathbf{P}(s,t) & = & \mathbf{P}(s,t)\mathbf{Q}(t) \\ \mathbf{P}(s,s) & = & \mathbf{I} \end{cases}$$

Proof

$$p_{ij}(s,t+h) = \sum_{k \in S} p_{ik}(s,t) p_{kj}(t,t+h) = \cdots = \sum_{k \in S} p_{ik}(s,t) \left[q_{kj}(t)h + o(h) \right] + p_{ij}(s,t)$$

- As in the time homogeneous case, the FDE decouple so that there is a separate set of simultaneous equations in $p_{ij}(\cdot,\cdot)$, for all $j \in S$, for each initial state $i \in S$.
- Further, again for each i, by summing these equations over all j, we see that one of the is always redundant.

The backward differential equations

Theorem

$$\frac{\partial}{\partial s} p_{ij}(s,t) = -\sum_{k \in S} q_{ik}(s) p_{kj}(s,t)$$

under the initial conditions $p_{ij}(t,t) = \delta_{ij}$.
In matrix form

$$\begin{cases} \frac{\partial}{\partial s}\mathbf{P}(s,t) & = & -\mathbf{Q}(s)\mathbf{P}(s,t) \\ \mathbf{P}(t,t) & = & \mathbf{I} \end{cases}$$

Proof

$$p_{ij}(s-h,t) = \sum_{k \in S} p_{ik}(s-h,s) p_{kj}(s,t) = \cdots = \sum_{k \in S} \left[q_{ik}(s-h)h + o(h) \right] p_{kj}(s,t) + p_{ij}(s,t)$$

$$\frac{p_{ij}(s,t) - p_{ij}(s-h,t)}{h} = -\sum_{k \in S} q_{ik}(s-h) p_{kj}(s,t) + \frac{o(h)}{h}$$

Holding times

Theorem

The **holding time** in any state $i \in S$ of a time inhomogeneous Markov process with transition rate matrix $\mathbf{Q}(s)$ is exponential distributed with mean

$$\frac{1}{-\int_s^t q_{ii}(u)du}$$

This is to say

$$p_{\overline{ii}}(s,t) = e^{\int_s^t q_{ii}(u)du}$$

where

$$p_{\overline{ii}}(s,t) = P(X(u) = i \text{for all } u \text{ such that } s \leqslant u \leqslant t | X(s) = i)$$

is the probability that the process starting in state i ate time s remains in state i ate least until t.

Integrated form of the Kolmogorov backward equations

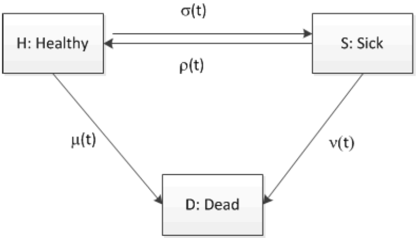
Integrated form of the Kolmogorov backward equations

$$\begin{aligned} p_{ij}(s,t) &= \sum_{k \neq i} \int_0^{t-s} p_{\overline{ii}}(s,s+w) q_{ik}(s+w) p_{kj}(s+w,t)dw, \quad j \neq i \\ p_{ii}(s,t) &= \sum_{k \neq i} \int_0^{t-s} p_{\overline{ii}}(s,s+w) q_{ik}(s+w) p_{ki}(s+w,t)dw + p_{\overline{ii}}(s,t) \end{aligned}$$

You get the backward equations by dierentiating the integral equations with respect to s.

Integrated form of the Kolmogorov backward equations

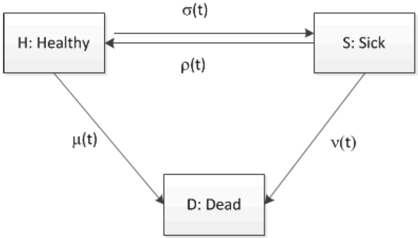
Example



Write down the integrate form of the backward equation for $p_{HD}(s, t)$

Integrated form of the Kolmogorov backward equations

Example



Write down the integrated form of the backward equation for $p_{HD}(s,t)$:

$$p_{HD}(s,t) = \int_0^{t-s} p_{\overline{HH}}(s,s+w)\sigma(s+w)p_{SD}(s+w,t)dw + \int_0^{t-s} p_{\overline{HH}}(s,s+w)\mu(s+w)dw$$

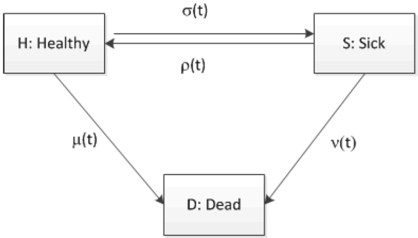
Integrated form of the Kolmogorov forward equations

Integrated form of the Kolmogorov forward equations

$$
\begin{aligned}
p_{ij}(s,t) &= \sum_{k \neq j} \int_0^{t-s} p_{ik}(s, t-w) q_{kj}(t-w) p_{\overline{jj}}(t-w, t)\, dw, \quad j \neq i \\
p_{ii}(s,t) &= \sum_{k \neq j} \int_0^{t-s} p_{ik}(s, t-w) q_{ki}(t-w) p_{\overline{ii}}(t-w, t) dw + p_{\overline{ii}}(s,t)
\end{aligned}
$$

Integrated form of the Kolmogorov backward equations

Example



Write down the integrate forms of the backward and forward equations for $p_{HS}(s,t)$

Integrated form of the Kolmogorov backward equations

Example

- Backward

$$\begin{aligned} p_{HS}(s,t) &= \int_0^{t-s} p_{\overline{HH}}(s,s+w)\sigma(s+w)p_{SS}(s+w,t)\,dw \\ &= \int_0^{t-s} e^{-\int_s^{s+w}(\sigma(u)+\mu(u))du}\sigma(s+w)p_{SS}(s+w,t)\,dw \end{aligned}$$

- Forward

$$\begin{aligned} p_{HS}(s,t) &= \int_0^{t-s} p_{HH}(s,t-w)\sigma(t-w)p_{\overline{SS}}(t-w,t)\,dw \\ &= \int_0^{t-s} p_{HH}(s,t-w)\sigma(t-w)e^{-\int_{t-w}^{t}(\rho(u)+\nu(u))du}\,dw \end{aligned}$$

Other notations

Other notations

- There are many different notations that can be used in Markov chains
- One that you should be familiar with, beacuse it is commonly used in survival models and life contingencies, is:

$$_t p_x^{ij} = P(X(x+t) = j | X(x) = i)$$

which denotes the parobability that a life in state i at age x is in state j at age $x+t$.
In this case

$$_t p_x^{\overline{ii}} = P(X(x+\tau) = i, \forall \tau \leqslant t | X(x) = i)$$

denotes the probability tha a life in state i at age x remains in state i until at least age $x+t$.
- Under this notation, the transition rates between states i and j at time x are usually denoted by μ_x^{ij}.

Other notations

Example

Consider a illness-death model with 3 states

1 - Healthy

2 - Sick

3 - Death

with transition rates μ_x^{ij}, $i, j = 1, 2, 3$.
Show from first principles that the forward differential equation for ${}_tp_x^{12}$ is

$$\frac{\partial}{\partial}{}_tp_x^{12} = {}_tp_x^{11}\mu_{x+t}^{12} - {}_tp_x^{12}\mu_{x+t}^{21} - {}_tp_x^{12}\mu_{x+t}^{23}$$

Noticing that

$$\begin{aligned} {}_{t+h}p_x^{12} &= {}_tp_x^{11}\,{}_hp_{x+t}^{12} + {}_tp_x^{12}\,{}_hp_{x+t}^{22} \quad \text{Chapman-Kolmogorov equations} \\ {}_hp_{x+t}^{12} &= h\,\mu_{x+t}^{12} + o(h) \quad \text{transition rates} \\ {}_hp_{x+t}^{22} &= 1 - h\mu_{x+t}^{21} - h\mu_{x+t}^{23} + o(h) \quad \text{transition rates} \end{aligned}$$

Other notations

In general

$$
\begin{aligned}
\frac{\partial}{\partial t}\,{}_t p_x^{ij} &= \sum_{k\in S} {}_t p_x^{ik}\,\mu_{x+t}^{kj} \\
&= \sum_{k\neq j} {}_t p_x^{ik}\,\mu_{x+t}^{kj} + {}_t p_x^{ij}\,\mu_{x+t}^{jj} \\
&= \sum_{k\neq j} {}_t p_x^{ik}\,\mu_{x+t}^{kj} + {}_t p_x^{ij}\left(-\sum_{k\neq j}\mu_{x+t}^{jk}\right) \\
&= \sum_{k\neq j}\left({}_t p_x^{ik}\,\mu_{x+t}^{kj} - {}_t p_x^{ij}\,\mu_{x+t}^{jk}\right)
\end{aligned}
$$