Master in Actuarial Sciences

Probability and Stochastic Processes

04/01/2019

Time allowed: Three hours

Instructions:

1. This paper contains 6 questions and comprises 3 pages including the title page.
2. Enter all requested details on the cover sheet.
3. You have 10 minutes of reading time. You must not start writing your answers until instructed to do so.
4. Number the pages of the paper where you are going to write your answers.
5. Attempt all 6 questions.
6. Begin your answer to each of the 6 questions on a new page.
7. Marks are shown in brackets. Total marks: 200.
8. Show calculations where appropriate.
9. An approved calculator may be used.
10. The distributed Formulary and the Formulae and Tables for Actuarial Examinations (the 2002 edition) may be used. **Note that, unless otherwise stated, the parametrisation used for the different distributions is that of the distributed Formulary.**

1. Losses, X, in hundreds of euros, on a certain line of business, are Gamma distributed with parameters θ and $\alpha = 3$, where θ is the outcome of a random variable Θ normally distributed with parameters $\mu = 5$ and $\sigma^2 = 10$. Find the unconditional expected loss and its standard deviation. [10]

2. Individual losses, X, given that a claim occurs, on a certain line of business are assumed to have cumulative distribution function given by

$$F_X(x) = \exp\left(-\left(\frac{x-\mu}{\theta}\right)^{-\alpha}\right), \qquad x > 0, \quad \alpha, \theta > 0, \quad x \geqslant \mu$$

(a) Identify the distribution of X and show that this distribution is max-stable. [10]

(b) Show that $\theta W^{-1/\alpha} + \mu$, where W is exponentially distributed with mean equal to one, is identically distributed to X. [05]

(c) Classify parameter μ, justifying. [05]

(d) Write the expression for the distribution function of $X - \mu$ and classify its parameter θ. [05]

(e) Consider now $\mu = 0$, $\theta = 4$ and $\alpha = 2$.

i) Determine, showing the calculations, the 95th percentile of X. [05]

ii) Compare the tail of this distribution with that of a Pareto with parameters $\alpha = 2$ and $\theta = 5$, according to the limit of the ratio of the two survival functions. [10]

iii) From experience, the company assumes a claim occurs with 10% probability and that no claims occur with 90% probability. What is the unconditional distribution of the individual loss, Y, assuming that whenever a claim occurs, the claim severity has distribution function $F_X(x)$? [10]

iv) Classify the random variable Y, defined in the previous question. [05]

3. An actuary considers a model for the losses on a certain line of business, in hundreds of euros, in the following way. For losses between 0 and 15, the probability density function is proportional to an Exponential distribution with mean 5. For losses higher than 15, the probability density function is proportional to a Pareto distribution with parameters $\alpha = 3$ and $\theta = 30$. The actuary builds the model in such way that the density function is continuous. Obtain the density function. [15]

4. Let X and Y be non-negative random variables whose joint survival distribution is

$$S_{X,Y}(x, y) = (e^x + e^y - 1)^{-1}, \qquad x, y > 0$$

(a) Show that X and Y are exponential distributions with mean equal to one and obtain the survival copula of X and Y. [10]

(b) Compute the index of upper tail dependence between X and Y. [10]

5. An insurance company has designed a Bonus-Malus system with five states. State 1: policies with 60% of the *a priori* premium; state 2: policies with 80% of the *a priori* premium; state 3: policies with 100% of the *a priori* premium; state 4: policies with 150% of the *a priori* premium; state 5: policies with 200% of the *a priori* premium. Transitions between states have the following rules:

- All policyholders start with a 20% discount of the *a priori* premium.
- If there are no claims in the current year, the policyholder will go to the maximum discount level next year.
- If there is a claim in the current year, the policyholder will move up one state next year, or remain in the maximum penalisation state.
- If there are two claims or more in the current year, the policyholder will move up two states or remain in the maximum penalisation state.

The probability of no claims in one year is 0.85 and the probability of having one claim in one year is 0.1. The *a priori* premium is 400 euros.

(a) Write down the transition probability matrix, and say, justifying, if it is regular. [10]

(b) What is the probability that a policyholder just entering the company will be in the maximum discount level in two years? What is the probability that she will be in the maximum penalisation level in two years? [10]

(c) What is the expected premium when the policy is renewed for the first time? And how much is it on the second renewal? [10]

(d) Explain why there is a limiting distribution and obtain it. [10]

(e) What is the expected premium paid by a policyholder in the long-run? [05]

(f) What is the expected number of years that a policyholder that has been in the company for a long time will take to return to the maximum discount level, after leaving it. [05]

(g) Using first step analysis, calculate the expected number of years a policyholder entering the company takes to visit for the first time the maximum penalisation level. [10]

6. Patients at a medical institution are classified as either "non-demented" or "demented". At follow-up at age x, the subjects will be in one of the following three states: "non-demented" (state 0), "demented" (state 1) or "dead" (state 2). At any possible age x a subject who was "non-demented" at age $x-s$, $s>0$, can be in any of the three states at age x. The transition from "non-demented" to "demented" is assumed irreversible.

Consider that the transition hazards, at age x, are

$$\begin{aligned}\mu_{01}(x) &= 0.005+0.003e^{0.015x}\\ \mu_{02}(x) &= 0.0075+0.001e^{0.01x}\\ \mu_{12}(x) &= 0.0075+0.003e^{0.04x}\end{aligned}$$

(a) Draw the transition graph and identify the generator matrix. [05]

(b) Determine ${}_tp_x^{00}$ and ${}_{10}p_{65}^{00}$. Interpret the meaning of ${}_{10}p_{65}^{00}$. [10]

(c) What is the probability that a demented individual aged 75 will die in the next 5 years? [10]

(d) Interpret the meaning of ${}_{10}p_{65}^{01}$ and give an integral expression for it, describing each element in the integral. Do not solve. [15]