LISBON SCHOOL OF ECONOMICS & MANAGEMENT

UNIVERSIDADE DE LISBOA

Master in Actuarial Sciences

Probability and Stochastic Processes

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1. [10]

$$X|\Theta=\theta \sim Gamma(\theta,\alpha=3) \qquad E[X|\Theta=\theta]=3\theta \text{ and } V[X|\Theta=\theta]=3\theta^2$$
$$\Theta \sim N(\mu=5,\sigma^2=10) \qquad E[\Theta]=5, V[\Theta]=10, E[\Theta^2]=\sigma^2+\mu^2=35$$

$$\begin{aligned} E[X] &= E[E(X|\Theta)]=E[3\Theta]=3E[\Theta]=15 \\ V[X] &= V[E(X|\Theta)]+E[V(X|\Theta)]=V[3\Theta]+E[3\Theta^2]=9V[\Theta]+3E[\Theta^2]=195 \\ \sqrt{V[X]} &= \sqrt{195}=13.9642 \end{aligned}$$

2. (a) X is distributed as a Fréchet extreme value distribution. [10]

Let $M_n=\max(X_1,\ldots,X_n)$, where X_i, $i=1,\ldots,n$ are i.i.d to X and n is fixed.

$$P(M_n\leqslant x) = [P(X\leqslant x)]^n=\left(\exp\left(-\left(\frac{x-\mu}{\theta}\right)^{-\alpha}\right)\right)^n=\exp\left(-n\left(\frac{x-\mu}{\theta}\right)^{-\alpha}\right)=\exp\left(-\left(\frac{x-\mu}{n^{1/\alpha}\theta}\right)^{-\alpha}\right)$$

The distribution of M_n is also a Frechét distribution with new parameter $\theta^*=n^{1/\alpha}\theta$, thus this distribution is max-stable.

(b) $W\sim Exp(1)$, $\qquad P(W>w)=e^{-x}$ [05]

$$P(\theta W^{-1/\alpha}+\mu\leqslant x) = P\left(W^{-1/\alpha}\leqslant\frac{x-\mu}{\theta}\right)=P\left(W>\left(\frac{\theta}{x-\mu}\right)^{\alpha}\right)=P\left(W>\left(\frac{x-\mu}{\theta}\right)^{-\alpha}\right)=e^{-\left(\frac{x-\mu}{\theta}\right)^{-\alpha}}=P(X\leqslant x)$$

(c) μ is a location parameter because the distribution of $X-\mu$ does not depended on μ: [05]

$$P(X-\mu\leqslant x)=P(X\leqslant x+\mu)=\exp\left(-\left(\frac{x+\mu-\mu}{\theta}\right)^{-\alpha}\right)=\exp\left(-\left(\frac{x}{\theta}\right)^{-\alpha}\right)$$

(d) $P(X-\mu\leqslant x)=\exp\left(-\left(\frac{x}{\theta}\right)^{-\alpha}\right)$. [05]

θ is a scale parameter for $X-\mu$, because the distribution of $\dfrac{X-\mu}{\theta}$ is independent of θ:

$$P\left(\frac{X-\mu}{\theta}\leqslant x\right)=P(X-\mu\leqslant\theta x)=\exp\left(-\left(\frac{\theta x}{\theta}\right)^{-\alpha}\right)=\exp\left(-(x)^{-\alpha}\right)$$

(e) i) [05]

$$q:F_X(q)=0.95 \quad\Leftrightarrow\quad q=\frac{4}{\sqrt{-\log(0.95)}}=17.6616$$

ii) $Z \sim Pareto(\alpha = 2, \theta = 5)$, $P(Z > x) = \left(\dfrac{5}{5+x}\right)^2$ [10]

$$\lim_{x\to\infty} \frac{S_X(x)}{S_Z(x)} = \lim_{x\to\infty} \frac{1-e^{-\left(\frac{x}{4}\right)^{-2}}}{\left(\frac{5}{5+x}\right)^2} \quad = \quad \lim_{x\to\infty} \frac{-2\,\frac{4^2}{x^3}\,e^{-\left(\frac{x}{4}\right)^{-2}}}{-2\,\frac{25}{(5+x)^3}} = \lim_{x\to\infty} \left(\frac{5+x}{x}\right)^3 \frac{16}{25} e^{-\left(\frac{x}{4}\right)^{-2}} = \frac{16}{25}$$

They have the same tail behaviour, according to the limit of the ratio of their survival functions.

iii) Defining $X^{(d)}$ such that $P(X^{(d)} = 0) = 1$, *i.e.* $F^{(d)}(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geqslant 0 \end{cases}$, then [10]

$$\begin{aligned} F_Y(y) &= 0.9F^{(d)}(y) + 0.1F_X(y) \\ &= 0.9 \begin{cases} 0, & x < 0 \\ 1, & x \geqslant 0 \end{cases} + 0.1 \begin{cases} 0, & x < 0 \\ e^{-\left(\frac{y}{4}\right)^{-2}}, & x \geqslant 0 \end{cases} = \begin{cases} 0, & x < 0 \\ 0.9 + 0.1e^{-\left(\frac{y}{4}\right)^{-2}}, & x \geqslant 0 \end{cases} \end{aligned}$$

iv) Y is a mixed random variable with discrete part $F^{(d)}(y)$ and continuous part $F_X(y)$. [05]

3. Let $X_1 \sim Exp(\theta = 5)$, with $f_{X_1}(x) = \dfrac{1}{5} e^{-\frac{x}{5}}$ and $X_2 \sim Pareto(\alpha = 3, \theta = 30)$, with $f_{X_2}(x) = \dfrac{3 \times 30^3}{(x+30)^4}$. [15]

$$\int_0^{15} \frac{1}{5} e^{-\frac{x}{5}}\, dx = 1 - e^{-3} \qquad \text{and} \qquad \int_{15}^{+\infty} \frac{3 \times 30^3}{(x+30)^4}\, dx = \left(\frac{30}{45}\right)^3$$

The spliced density function is:

$$f(x) = \begin{cases} p\,\dfrac{1}{5} \times \dfrac{e^{-\frac{x}{5}}}{1-e^{-3}}, & 0 < x < 15 \\[2ex] (1-p)\left(\dfrac{45}{30}\right)^3 \dfrac{3 \times 30^3}{(x+30)^4}, & x > 15 \end{cases}$$

To obtain continuity:

$$p\,\frac{1}{5} \times \frac{e^{-\frac{15}{5}}}{1-e^{-3}} = (1-p)\left(\frac{45}{30}\right)^3 \frac{3 \times 30^3}{(15+30)^4} \quad \Longleftrightarrow \quad p = 0.864164$$

Thus

$$f(x) = \begin{cases} 0.864164 \times \dfrac{1}{5}\dfrac{e^{-\frac{x}{5}}}{1-e^{-3}}, & 0 < x < 15 \\[2ex] 0.135836 \times \left(\dfrac{45}{30}\right)^3 \dfrac{3 \times 30^3}{(x+30)^4}, & x > 15 \end{cases}$$

4. (a) [10]

$$\begin{aligned} P(X > x) &= P(X > x, Y > 0) = \left(e^x + 1 - 1\right)^{-1} = e^{-x} \quad \Longrightarrow X \sim Exp(1) \\ P(Y > y) &= P(X > 0, Y > y) = \left(1 + e^y - 1\right)^{-1} = e^{-y} \quad \Longrightarrow Y \sim Exp(1) \end{aligned}$$

The survival copula $\overline{C}(u, v)$ is such that $\overline{C}(1 - F_X(x), 1 - F_Y(y)) = P(X > x, Y > y)$.
Since $P(X > x, Y > y) = \left((1 - F_X(x))^{-1} + (1 - F_Y(y))^{-1} - 1)\right)^{-1}$, we obtain $\overline{C}(u, v) = \left(u^{-1} + v^{-1} - 1\right)^{-1}$.

(b) [10]

$$\lambda_U = \lim_{u\to1} \frac{\overline{C}(1-u, 1-u)}{1-u} = \lim_{u\to1} \frac{\left(\frac{1}{1-u} + \frac{1}{1-u} - 1\right)^{-1}}{1-u} = \cdots = \frac{1}{2}$$

5. (a) All states comunicate and are aperiodic, thus, being finite, the chain is regular. [10]

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{r} 1 \\ 2 \\ P = 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0.85 & 0.1 & 0.05 & 0 & 0 \\ 0.85 & 0 & 0.1 & 0.05 & 0 \\ 0.85 & 0 & 0 & 0.1 & 0.05 \\ 0.85 & 0 & 0 & 0 & 0.15 \\ 0.85 & 0 & 0 & 0 & 0.15 \end{pmatrix} \end{array}$$

(b) [10]

$$P_{21}^{(2)} = (0.85, 0, 0.1, 0.05, 0) \cdot (0.85, 0.85, 0.85, 0.85, 0.85) = 0.85$$
$$P_{25}^{(2)} = (0.85, 0, 0.1, 0.05, 0) \cdot (0., 0., 0.05, 0.15, 0.15) = 0.0125$$

(c) The transition probabilities from state two in one step are (second line of P) $P_{2i} = (0.85, 0, 0.1, 0.05, 0)$, and the transition probabilities from state 2 in two steps are (second line of P^2) $P_{2i}^{(2)} = (0.85, 0.085, 0.0425, 0.01, 0125)$. [10]

Exp. premium on the 1st renewal: $0.85 \times 240 + 0.1 \times 400 + 0.05 \times 600 = 274$

Exp. premium on the 2nd renewal: $0.85 \times 240 + 0.085 \times 320 + 0.0425 \times 400 + 0.01 \times 600 + 0.0125 \times 800 = 264.2$

(d) The liminting distribution exists because the chain is regular, since it is irreducible, finite and aperiodic. Solving $\boldsymbol{\pi} P = \boldsymbol{\pi}$ for $\boldsymbol{\pi}$, with $\pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 = 1$,we obtain [10]

$$\boldsymbol{\pi} = (0.85, 0.085, 0.051, 0.00935, 0.00465)$$

(e) In the long-run, the expected premium is [05]

$$(0.85, 0.085, 0.051, 0.00935, 0.00465) \cdot (240, 320, 400, 600, 800) = 260.93$$

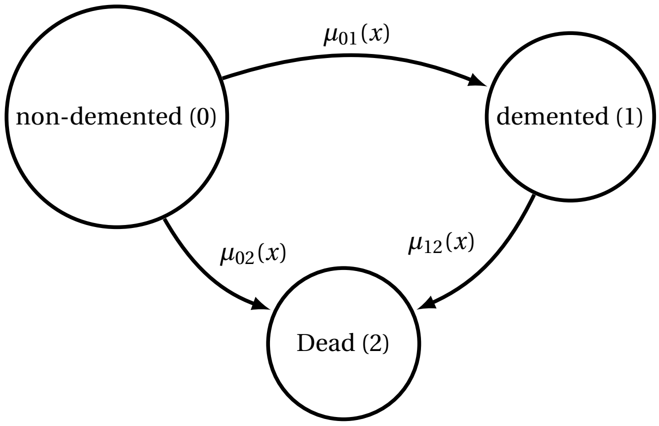
(f) $m_1 = \dfrac{1}{\pi_1} = \dfrac{1}{0.85} = 1.1765$ [05]

(g) Let $T = \min\{n \geqslant 0 : X_n = 5\}$, the number of years a policyholder takes to visit state 5 for the first time. And let $v_i = E[T|X_0 = i]$, $i = 1, \ldots, 5$. The quantity asked for is v_2. Using first step analysis, we have the following system of equations: [10]

$$\begin{cases} v_1 &= 1 + 0.85 v_1 + 0.1 v_2 + 0.05 v_3 \\ v_2 &= 1 + 0.85 v_1 + 0.1 v_3 + 0.05 v_4 \\ v_3 &= 1 + 0.85 v_1 + 0.1 v_4 \\ v_4 &= 1 + 0.85 v_1 \end{cases}$$

Solving the system we obtain $v_2 = 249.46$ years.

6. (a) [05]



$$Q = \begin{bmatrix} -(\mu_{01}(x) + \mu_{02}(x)) & \mu_{01}(x) & \mu_{02}(x) \\ 0 & -\mu_{12}(x) & \mu_{12}(x) \\ 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} -0.0125 - 0.003e^{0.015x} - 0.001e^{0.01x} & 0.005 + 0.003e^{0.015x} & 0.0075 + 0.001e^{0.01x} \\ 0 & -0.0075 - 0.003e^{0.04x} & 0.0075 + 0.003e^{0.04x} \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Since ${}_tp_x^{10} = {}_tp_x^{20} = 0$: [10]

$$\begin{aligned} {}_tp_{65}^{00} &= {}_tp_{65}^{\overline{00}} = \exp\left(-\int_{65}^{65+t}\left(\mu_{01}(x)+\mu_{02}(x)\right)\,dx\right) \\ &= \exp\left(-\left(0.0125\,t+0.2\,e^{0.015\times 65}\left(e^{0.015\,t}-1\right)+0.1\,e^{0.01\times 65}\left(e^{0.01\,t}-1\right)\right)\right) \end{aligned}$$

$$ {}_{10}p_{65}^{00} = \exp\left(-\left(0.0125\times 10+0.2\,e^{0.015\times 65}\left(e^{0.015\times 10}-1\right)+0.1\,e^{0.01\times 65}\left(e^{0.01\times 10}-1\right)\right)\right) = 0.793775$$

${}_{10}p_{65}^{00}$ is the probability that an individual aged 65 that is non demented is still non demented at the age of 75.

(c) [10]

$$\begin{aligned} {}_5p_{75}^{12} &= 1-{}_5p_{75}^{11} = 1-{}_5p_{75}^{\overline{11}} = 1-e^{-\int_{75}^{80}\mu_{12}(x)\,dx} = 1-e^{-\int_{75}^{80}\left(0.0075+0.003e^{0.04x}\right)\,dx} \\ &= 1-e^{-0.0075\times 5-\frac{0.003}{0.04}\left(e^{0.04\times 80}-e^{0.04\times 75}\right)} = 0.309973 \end{aligned}$$

(d) ${}_{10}p_{65}^{01}$ is the probability that an individual aged 65 that is non demented will become demented before the age of 75. [15]

$$ {}_{10}p_{65}^{01} = \int_0^{10} {}_wp_{65}^{00}\,\mu_{01}(65+w)\ {}_{10-w}p_{65+w}^{11}\,dw$$

where

$$\begin{aligned} {}_wp_{65}^{00} &= \exp\left(-\left(0.0125\,w+0.2\,e^{0.015\times 65}\left(e^{0.015\,w}-1\right)+0.1\,e^{0.01\times 65}\left(e^{0.01\,w}-1\right)\right)\right) \\ \mu_{01}(65+w) &= 0.005+0.003\,e^{0.015\,(65+w)} \\ {}_{10-w}p_{65+w}^{11} &= {}_{10-w}p_{65+w}^{\overline{11}} = \exp\left(-\int_{65+w}^{65+w+(10-w)}\mu_{12}(x)\,dx\right) = \exp\left(-\int_{65+w}^{75}\left(0.0075+0.003\,e^{0.04\,x}\right)\,dx\right) \end{aligned}$$