Master in Actuarial Sciences

Probability and Stochastic Processes

31/01/2019

Time allowed: Three hours

Instructions:

1. This paper contains 8 questions and comprises 3 pages including the title page.
2. Enter all requested details on the cover sheet.
3. You have 10 minutes of reading time. You must not start writing your answers until instructed to do so.
4. Number the pages of the paper where you are going to write your answers.
5. Attempt all 8 questions.
6. Begin your answer to each of the 8 questions on a new page.
7. Marks are shown in brackets. Total marks: 200.
8. Show calculations where appropriate.
9. An approved calculator may be used.
10. The distributed Formulary and the Formulae and Tables for Actuarial Examinations (the 2002 edition) may be used. **Note that, unless otherwise stated, the parametrisation used for the different distributions is that of the distributed Formulary.**

1. Actuaries of an insurance company assume that the mean number of claims per year in the motor insurance line of business is 0.18, and that the variance of the number of claims per year is also 0.18. What would you say about the assumptions on the values of the mean and the variance if, in a portfolio of 1000 policies, more than 218 claims had been registered at the end of an year? [10]

2. Let $X \sim Gamma(\alpha, \theta)$. Show that the moment generating function of X is $M_X(t) = (1-\theta t)^{-\alpha}$. [10]

3. Consider that the claim amount, given that a claim occurs, in thousands of euros, is a random variable X with probability density function given by:

$$f_X(x) = 12xe^{-4x} + xe^{-2x}, \quad x \geqslant 0.$$

(a) Describe X as the mixture of two random variables, identifying them. [05]

(b) Obtain the moment generating function of X. [05]

(c) Show that the survival function of X is $S_X(x) = 3\left(x + \frac{1}{4}\right)e^{-4x} + \frac{1}{2}\left(x + \frac{1}{2}\right)e^{-2x}$. [10]

(d) Determine the probability density function of the equilibrium distribution of X. [05]

(e) Using the ratio of the survival distributions, compare the tail of this distribution with that of a Gamma with parameters $\alpha = 2$ and $\lambda = \frac{1}{\theta}$, for varying values of λ. [10]

(f) When a claim occurs, the insurance company applies a deductible of 150 euros (0.15 thousands euros), that is, it covers only the amount of the claim in excess of 150 euros, and if the claim amount is lower than 150 euros, the insurance company does not incur any payment. Define the insurer loss random variable and provide its distribution function. [10]

4. Consider random variables X, Y, with the following joint density function [15]

$$f(x,y) = \begin{cases} 2e^{-(x+y)}, & \text{for } 0 < \frac{x}{3} < y < 3x \\ \\ 0, & \text{otherwise} \end{cases},$$

where X is unlimited. Find the probability density function of the random variable $Z = Y/X$.

5. Let X and Y be random variables with a joint distribution function given by

$$H_\alpha(x,y) = \exp\left[-\left(e^{-\alpha x} + e^{-\alpha y}\right)^{1/\alpha}\right], \quad \alpha \geqslant 1$$

(a) Show that the marginal distribution functions of X and Y are $F_X(x) = e^{-e^{-x}}$ and $F_Y(y) = e^{-e^{-y}}$ and identify them. [05]

(b) Show that the copula of X and Y belongs to the Gumbel-Hougaard Archimedean family of copulas, *i.e.* [05]

$$C_\alpha(u_1, u_2) = \exp\left(-\left[(-\ln u_1)^\alpha + (-\ln u_2)^\alpha\right]^{1/\alpha}\right)$$

(c) Show that the copula of X and Y belongs to a max-stable family of copulas and explain in words what this property means. [10]

6. Bonds on a certain market are rated once every year at the same time by a rating agency. The rating agency uses a Markov chain to model the credit rating evolution of bonds, considering four states, ratings A, B, C and default D. The probability transition matrix is as follows

$$P = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0.7 & 0.2 & 0.1 & 0 \\ 0.1 & 0.7 & 0.1 & 0.1 \\ 0 & 0.2 & 0.7 & 0.1 \\ 0 & 0 & 0.1 & 0.9 \end{pmatrix} \end{array}$$

(a) Draw the transition graph of the chain and say, justifying, if the chain is regular. [05]

(b) What is the probability that a bond currently rated as A will default in two years? [05]

(c) What is the probability that a bond currently rated A that is being downgraded to C will ever be rated A again. Justify. [05]

(d) Write down the system of equations that allows you to obtain the expected number of years a bond will visit rate A before defaulting. Do not solve. [05]

(e) In the long-run, what is the percentage of bonds rated as A? And the percentage of bonds in default? [15]

7. A taxi company has one mechanic who replaces fuel pumps when they fail. Assume that the waiting time in days until a fuel pump fails is exponentially distributed with mean 300; the company has 1000 cars; and the repair time for each car is exponentially distributed with expected repair time 1/4 days.

(a) Draw the transition graph and specify the generator matrix. [05]

(b) Using Kolmogorov's forward differential equations, find the probability that a car with a currently functional fuel pump, will have the fuel pump broken at time t. [15]

(c) What is the expected number of cars with a broken fuel pump after the taxi company has been working for a long time? [10]

8. A given model for the retirement payment scheme depends on the employment state of the individual. A person can be employed (state E), unemployed (state U) or dead (state D). An employed individual can become unemployed, and vice-versa. The transition rates for a person aged x are:

$$
\begin{aligned}
\mu_{EU}(x) &= 0.01 + 0.001e^{0.025x} \\
\mu_{UE}(x) &= 0.005 + 0.05e^{-0.01x} \\
\mu_{ED}(x) = \mu_{UD}(x) &= 0.0005
\end{aligned}
$$

Time is expressed in years.

(a) Draw the transition graph and specify the appropriate transition rate matrix. [05]

(b) What is the probability that an unemployed individual aged 50 will remain unemployed until the age of 65? [10]

(c) Given that an employed individual has made a transition out of state E at age x, find the probability that it was into state D. What is that probability for an individual aged 60? [05]

(d) Suppose that an individual is employed at age x, with $x < T - 0.25$. Provide and integral expression for the probability that she is unemployed at time T and has been unemployed for less than 3 months, expressed in terms of $P_{EE}(x, t)$. Clearly specify all elements in the integral. Do not solve. [15]