LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

Master in Actuarial Sciences

Probability and Stochastic Processes

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1. N : number of claims per year, $E[N] = V[N] = 0.18$. [10]

$$P(\overline{X} > 0.218) = P\left(\frac{\overline{X} - 0.18}{\sqrt{0.18/1000}} > \frac{0.218 - 0.18 - \frac{0.5}{1000}}{\sqrt{0.18/1000}}\right) \underbrace{\approx}_{C.L.T.} 1 - \Phi(2.795) = 1 - 0.9974 = 0.0026$$

Thus, it is unlikely to observe more than 218 claims in one year, in a portfolio of 1000 policies, with these assumptions for the mean and variance of the number of claims. With these observations, these assumptions should be revised.

2. $f_X(x) = \dfrac{x^{\alpha-1}\, e^{-x/\theta}}{\theta^\alpha \Gamma(\alpha)}$ [10]

$$\begin{aligned} M_X(t) &= E\left[e^{tX}\right] = \int_0^{+\infty} e^{tx} \frac{x^{\alpha-1}\, e^{-x/\theta}}{\theta^\alpha \Gamma(\alpha)}\, dx = \int_0^{+\infty} \frac{x^{\alpha-1}\, e^{-x(1/\theta - t)}}{\theta^\alpha \Gamma(\alpha)}\, dx \\ &= \theta^{-\alpha} \left(\frac{1-\theta t}{\theta}\right)^{-\alpha} \int_0^{+\infty} \frac{x^{\alpha-1}\, e^{-x\left(\frac{1-\theta t}{\theta}\right)}}{\left(\frac{\theta}{1-\theta t}\right)^\alpha \Gamma(\alpha)}\, dx = (1-\theta t)^{-\alpha} \end{aligned}$$

3. (a) X is the mixture of $X_1 \sim Gamma\left(\frac{1}{4}, 2\right)$ and $X_2 \sim Gamma\left(\frac{1}{2}, 2\right)$ as follows [05]

$$f_X(x) = \frac{3}{4} f_{X_1}(x) + \frac{1}{4} f_{X_2}(x) = \frac{3}{4} \frac{4^2 x e^{-4x}}{\Gamma(2)} + \frac{1}{4} \frac{2^2 x e^{-2x}}{\Gamma(2)} = 12 x e^{-4x} + x e^{-2x}$$

(b) [05]

$$M_X(t) = \frac{3}{4} M_{X_1}(t) + \frac{1}{4} M_{X_2}(t) = \frac{3}{4}\left(1 - \frac{t}{4}\right)^{-2} + \frac{1}{4}\left(1 - \frac{t}{2}\right)^{-2}$$

(c) [10]

$$\begin{aligned} S_X(x) &= \int_x^{+\infty} f_X(t)\, dt = \int_x^{+\infty} \left(12 x e^{-4x} + x e^{-2x}\right) dt \\ &= 12\left(\left[t\frac{e^{-4t}}{-4t}\right]_x^{+\infty} - \left[\frac{e^{-4t}}{4^2}\right]_x^{+\infty}\right) + \left[t\frac{e^{-2t}}{-2}\right]_x^{+\infty} - \left[\frac{e^{-2t}}{2^2}\right]_x^{+\infty} \\ &= \frac{12}{4} x e^{-4x} + \frac{12 e^{-4x}}{16} + \frac{x e^{-2x}}{2} + \frac{e^{-2x}}{4} = 3\left(x + \frac{1}{4}\right) e^{-4x} + \frac{1}{2}\left(x + \frac{1}{2}\right) e^{-2x} \end{aligned}$$

(d) $f_X(x) = \dfrac{3}{4} f_{X_1}(x) + \dfrac{1}{4} f_{X_2}(x)$, with [05]

$X_1 \sim Gamma\left(\frac{1}{4},2\right)$, $E[X_1] = \dfrac{1}{4} \times 2 = \dfrac{1}{2}$, and $X_2 \sim Gamma\left(\frac{1}{2},2\right)$, $E[X_2] = \dfrac{1}{2} \times 2 = 1$

$$E[X] = \int_0^{+\infty} x\, f_X(x)\, dx = \frac{3}{4}\int_0^{+\infty} x\, f_{X_1}(x)\, dx + \frac{1}{4}\int_0^{+\infty} x\, f_{X_2}(x)\, dx = \frac{3}{4}E[X_1] + \frac{1}{4}E[X_2] = \frac{3}{4} \times \frac{1}{2} + \frac{1}{4} = \frac{5}{8}$$

$$f_e(x) = \frac{S_X(x)}{E[X]} = \frac{8}{5}S_X(x) = \frac{24}{5}\left(x + \frac{1}{4}\right)e^{-4x} + \frac{4}{5}\left(x + \frac{1}{2}\right)e^{-2x}$$

(e) Let $X_3 \sim Gamma\left(\alpha = 2, \lambda = \dfrac{1}{\theta}\right)$, $f_{X_3}(x) = \theta^2 x e^{-\theta x}$ [10]

$$\lim_{x\to\infty} \frac{S_X(x)}{S_{X_3}(x)} = \lim_{x\to\infty} \frac{f_X(x)}{f_{X_3}(x)} = \lim_{x\to\infty} \frac{12\, x\, e^{-4x} + x\, e^{-2x}}{\theta^2 x e^{-\theta x}} = \begin{cases} +\infty, & \theta > 2 \Leftrightarrow \lambda < \frac{1}{2} \\ \theta^{-2}, & \theta = 2 \Leftrightarrow \lambda = \frac{1}{2} \\ 0, & \theta < 2 \Leftrightarrow \lambda > \frac{1}{2} \end{cases}$$

If $\lambda < \frac{1}{2}$, X is heavier tailed than X_3, if $\lambda = \frac{1}{2}$, X and X_3 have the same tail behaviour, if $\lambda > \frac{1}{2}$, X is lighter tailed than X_3.

(f) [10]

$$Y = \begin{cases} 0, & X < 0.15 \\ X - 0.15, & X \geqslant 0.15 \end{cases} = \max(0, X - 15),$$

$P(X < 0.15) = 1 - S_X(0.15) = 0.10066$, and $P(X - 0.15 \leqslant y) = P(X \leqslant y + 0.15) = 1 - S_X(y + 0.15)$, thus

$$F_Y(y) = P(y \leqslant y) = \begin{cases} 0, & y < 0 \\ P(X \leqslant 0.15), & y = 0 \\ P(X - 0.15 \leqslant y), & y > 0 \end{cases} = \begin{cases} 0, & y < 0 \\ 0.10066, & y = 0 \\ 1 - 3\left(y + 0.15 + \frac{1}{4}\right)e^{-4(y+0.15)} - \frac{1}{2}\left(y + 0.15 + \frac{1}{2}\right)e^{-2(y+0.15)}, & y > 0 \end{cases}$$

4. We have that [15]

$$f_Z(z) = \frac{d}{dz}F_Z(z) \qquad \text{and} \qquad P(Z \leqslant z) = P\left(\frac{Y}{X} \leqslant z\right) = \int_0^{+\infty} P(Y \leqslant zx | X = x) f_X(x)\, dx$$

Hence

$$\begin{aligned} f_Z(z) &= \frac{d}{dz}\int_0^{+\infty} P(Y \leqslant zx|X = x) f_X(x)\, dx = \int_0^{+\infty} \frac{d}{dz}P(Y \leqslant zx|X = x) f_X(x)\, dx \\ &= \int_0^{+\infty} x\, f_{Y|X=x}(x, zx) f_X(x)\, dx \end{aligned}$$

where $f_{Y|X=x}(x,y) = \dfrac{f(x,y)}{f_X(x)}$, thus $f_{Y|X=x}(x,xz) = \dfrac{f(x,xz)}{f_X(x)}$ and hence $f_{Y|X=x}(x,zx) f_X(x) = f(x,zx)$:

$$\begin{aligned} f_Z(z) &= \int_0^{+\infty} x\, f(x,zx)\, dx = \int_0^{+\infty} x\, 2\, e^{-(x+zx)}\, dx \\ &= 2\int_0^{+\infty} x\, e^{-x(1+z)}\, dx = 2\left[\frac{x}{-(1+z)}e^{-x(1+z)}\right]_0^{+\infty} - 2\int_0^{+\infty} \frac{e^{-x(1+z)}}{-(1+z)}\, dx \\ &= 2 \times 0 + 2\left[-\frac{e^{-x(1+z)}}{(1+z)^2}\right]_0^{+\infty} = \frac{2}{(1+z)^2} \end{aligned}$$

Thus

$$f_Z(z) = \begin{cases} \dfrac{2}{(1+z)^2}, & \dfrac{1}{3} < z < 3 \\ 0, & \text{otherwise} \end{cases}$$

5. (a) $F_X(x) = P(X \leqslant x, Y \leqslant +\infty) = e^{-\left(e^{-\alpha x}\right)^{1/\alpha}} = e^{-e^{-x}}$ and $F_Y(y) = P(X \leqslant +\infty, Y \leqslant y) = e^{-\left(e^{-\alpha y}\right)^{1/\alpha}} = e^{-e^{-y}}$, thus $F_X(x)$ and $F_Y(y)$ are standard Gumbel extreme value distributions. [05]

(b) [05]

$$\begin{aligned} C(u_1,u_2) &= C(F_X(x), F_Y(y)) = C\left(e^{-e^{-x}}, e^{-e^{-y}}\right) = \exp\left(-\left[(-\ln e^{-e^{-x}})^\alpha + (-\ln e^{-e^{-y}})^\alpha\right]^{1/\alpha}\right) \\ &= \exp\left(-\left[(e^{-x})^\alpha + (e^{-y})^\alpha\right]^{1/\alpha}\right) = \exp\left[-\left(e^{-\alpha x} + e^{-\alpha y}\right)^{1/\alpha}\right] = P(X \leqslant x, Y \leqslant y) \end{aligned}$$

(c) A copula C, between X and Y, is max-stable if the copula associating the maximum of X and the maximum of Y belongs to the same copula family of C. [10]

Let $M_{n,X} = \max(X_1, \ldots, X_n)$ and $M_{n,Y} = \max(Y_1, \ldots, Y_n)$, where $X_1, \ldots, X_n$ are i.i.d. to X and $Y_1, \ldots, Y_n$ are i.i.d. to Y, with n fixed. Then, $F_{M_{n,X}}(x) = [F_X(x)]^n$, $F_{M_{n,Y}}(y) = \left[F_Y(y)\right]^n$, and $P(M_{n,X} \leqslant x, M_{n,Y} \leqslant y) = \left[P(X \leqslant x, Y \leqslant y)\right]^n = \left[C(F_X(x), F_Y(y))\right]^n$.

Thus, the copula is max-stable if $C(F_{M_{n,X}}(x), F_{M_{n,Y}}(y)) = \left[C(F_X(x), F_Y(y))\right]^n$, *i.e* if $C([F_X(x)]^n, [F_Y(y)]^n) = \left[C(F_X(x), F_Y(y))\right]^n$.

$$\begin{aligned}
[C(u_1, u_2)]^n &= \left[\exp\left(-\left[(-\ln u_1)^\alpha + (-\ln u_2)^\alpha\right]^{1/\alpha}\right)\right]^n = \exp\left(-n\left[(-\ln u_1)^\alpha + (-\ln u_2)^\alpha\right]^{1/\alpha}\right) \\
&= \exp\left(-\left[(-n\ln u_1)^\alpha + (-n\ln u_2)^\alpha\right]^{1/\alpha}\right) = \exp\left(-\left[(-\ln u_1^n)^\alpha + (-\ln u_2^n)^\alpha\right]^{1/\alpha}\right) = C(u_1^n, u_2^n)
\end{aligned}$$

6. (a) The chain is finite, irreducible and aperiodic, so it is regular. [05]

(b) $P^{(2)}_{AD} = (0.7, 0.2, 0.1, 0) \cdot (0, 0.1, 0.1, 0.9) = 0.03$ [05]

(c) The chain is finite and irreducible, so all states are recurrent. Hence, the probability that a bond currently rate A, that is being downgrade to C, will ever be rated A again is 1, $f_{AA} = 1$. [05]

(d) Let w_i, with $i = A, B, C, D$, be the expected number of years a bond will visit rate A before defaulting, starting at state i. The quantity asked for is w_A. Using first step analysis, we obtain w_A as the solution of the following system: [05]

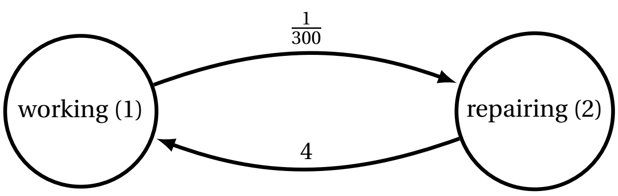
$$\begin{cases}
w_A &= 1 + 0.7\,w_A + 0.2\,w_B + 0.1\,w_C \\
w_B &= 0.1\,w_A + 0.7\,w_B + 0.1\,w_C \\
w_C &= 0.2\,w_B + 0.7\,w_C
\end{cases}$$

(e) The chain is regular, so there exists the limit distribution which can be obtained by solving the system $\boldsymbol{\pi} P = \boldsymbol{\pi}$ for $\boldsymbol{\pi}$, with $\pi_A + \pi_B + \pi_C + \pi_D = 1$: [15]

$$\boldsymbol{\pi} P = \boldsymbol{\pi} \Longleftrightarrow \begin{cases}
\pi_A &= 0.7\pi_A + 0.1\pi_B \\
\pi_B &= 0.2\pi_A + 0.7\pi_B + 0.2\pi_C \\
\pi_C &= 0.1\pi_A + 0.1\pi_B + 0.7\pi_C + 0.1\pi_D \\
\pi_D &= 0.1\pi_B + 0.1\pi_C + 0.9\pi_D
\end{cases} \Longleftrightarrow \begin{cases}
\pi_A &= \frac{2}{28} = 0.07143 \\
\pi_B &= \frac{6}{28} = 0.2143 \\
\pi_C &= \frac{7}{28} = 0.25 \\
\pi_D &= \frac{13}{28} = 0.4643
\end{cases}$$

In the long-run, the percentage of bonds rated A is 7,14% and the percentage of bonds in default is 46,43%.

7. (a)



$$Q = \begin{bmatrix} -\frac{1}{300} & \frac{1}{300} \\ 4 & -4 \end{bmatrix}$$ [05]

(b) The probability that a car with a currently functional fuel pump, will have the fuel pump broken at time t is $p_{12}(t)$. [15]

From the Kolmogorov's forward differential equations and since $p_{11}(t)+p_{12}(t)=1$, we have:

$$
\begin{aligned}
& \frac{d}{dt}p_{12}(t) = p_{12}(t)q_{22} + p_{11}(t)q_{12} \qquad \Longleftrightarrow \\
\Longleftrightarrow \qquad & \frac{d}{dt}p_{12}(t) = -4p_{12}(t) + \frac{1}{300}p_{11}(t) \\
\Longleftrightarrow \qquad & \frac{d}{dt}p_{12}(t) = -4p_{12}(t) + \frac{1}{300}(1-p_{12}(t)) \\
\Longleftrightarrow \qquad & \frac{d}{dt}p_{12}(t) + \left(4+\frac{1}{300}\right)p_{12}(t) = \frac{1}{300} \\
\Longleftrightarrow \qquad & e^{\left(4+\frac{1}{300}\right)t}\frac{d}{dt}p_{12}(t) + e^{\left(4+\frac{1}{300}\right)t}\left(4+\frac{1}{300}\right)p_{12}(t) = e^{\left(4+\frac{1}{300}\right)t}\frac{1}{300} \\
\Longleftrightarrow \qquad & \frac{d}{dt}\left(e^{\left(4+\frac{1}{300}\right)t}p_{12}(t)\right) = e^{\left(4+\frac{1}{300}\right)t}\frac{1}{300} \\
\Longleftrightarrow \qquad & e^{\left(4+\frac{1}{300}\right)t}p_{12}(t) = e^{\left(4+\frac{1}{300}\right)t}\frac{1}{300}\frac{1}{4+\frac{1}{300}} + C \\
\Longleftrightarrow \qquad & p_{12}(t) = \frac{1}{300}\frac{1}{4+\frac{1}{300}} + Ce^{-\left(4+\frac{1}{300}\right)t} \\
\Longleftrightarrow \qquad & p_{12}(t) = \frac{1}{1201} + Ce^{-\left(4+\frac{1}{300}\right)t}
\end{aligned}
$$

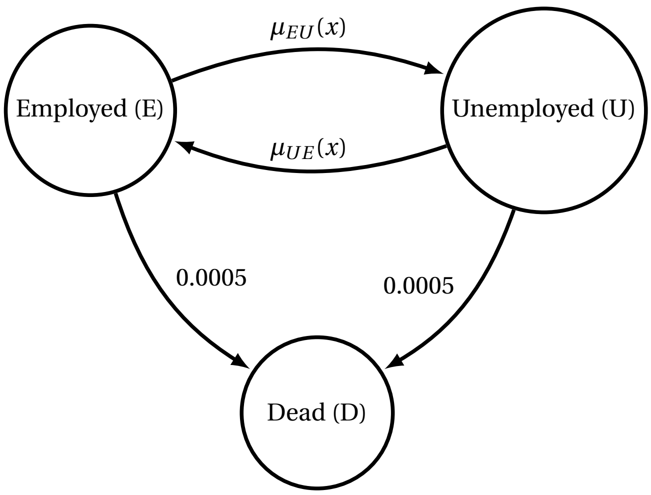
Using the initial condition $p_{12}(0)=0$, we obtain $C=-\frac{1}{1201}$. Thus $p_{12}(t)=\frac{1}{1201}-\frac{1}{1201}e^{-\left(4+\frac{1}{300}\right)t}$.

(c) The limiting distribution is given by the solution of $\boldsymbol{\pi}Q=\mathbf{0}$: [10]

$$
\begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix}\begin{bmatrix} -\frac{1}{300} & \frac{1}{300} \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \quad \Longleftrightarrow \quad -\frac{1}{300}\pi_1 + 4\pi_2 = 0 \quad \Longleftrightarrow \quad \pi_2 = \frac{1}{1200}\pi_1
$$

Since $\pi_1+\pi_2=1$, we obtain $\begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} = \begin{bmatrix} \frac{1200}{1201} & \frac{1}{1201} \end{bmatrix}$. For a fleet of 1000 cars, the expected number of cars with a broken fuel pump in the long run is $1000/1201 = 0.8326395$.

8. (a) [05]



$$
Q(x) = \begin{array}{c} \\ E \\ U \\ D \end{array}
\begin{array}{c}
\begin{array}{ccc} E & U & D \end{array} \\
\begin{pmatrix} -\mu_{EU}(x)-\mu_{ED}(x) & \mu_{EU}(x) & \mu_{ED}(x) \\ \mu_{UE}(x) & -\mu_{UE}(x)-\mu_{UD}(x) & \mu_{ED}(x) \\ 0 & 0 & 0 \end{pmatrix}
\end{array}
$$

$$
= \begin{array}{c} \\ E \\ U \\ D \end{array}
\begin{array}{c}
\begin{array}{ccc} E & U & D \end{array} \\
\begin{pmatrix} -0.0105-0.01e^{0.025x} & 0.01+0.01e^{0.025x} & 0.0005 \\ 0.005+0.05e^{0.01x} & -0.0055-0.05e^{-0.01x} & 0.0005 \\ 0 & 0 & 0 \end{pmatrix}
\end{array}
$$

(b) [10]

$$p_{\overline{UU}}(x,t) = \exp\left(-\int_x^t (\mu_{UE}(x)+\mu_{UD})\,dx\right) = e^{-(0.0055\,(t-x)+5(e^{-0.01\,t}-e^{-0.01\,x})}$$

$$p_{\overline{UU}}(50,65) = e^{-(0.0055\times 15+5(e^{-0.01\times 65}-e^{-0.01\times 50})} = e^{-0.5049244} = 0.6035512$$

(c) [05]

The probability is $\dfrac{\mu_{ED}(x)}{\mu_{EU}(x)+\mu_{ED}(x)} = \dfrac{0.0005}{0.0105+0.001\,e^{0.025\,x}}$.

For a person aged 60, this probability is $\dfrac{0.0005}{0.0105+0.001\,e^{0.025\times 60}} = 0.0334$.

(d) [15]

$$\begin{aligned}
&P(X(T)=U \text{ and has been unemployed for less than a quarter of a year} \,|\, X(x)=E) = \\
&= \int_{T-0.25}^{T} p_{EE}(x,w)\mu_{EU}(w)p_{\overline{UU}}(w,T)\,dw \\
&= \int_{T-0.25}^{T} p_{EE}(x,w)\left(0.01+0.01e^{0.025\,x}\right)e^{-\left(0.0055(T-w)+5\left(e^{0.01\,T}-e^{0.01\,w}\right)\right)}\,dw
\end{aligned}$$