

Probability and Stochastic Processes

Master in Actuarial Sciences

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General Notions of Stochastic Processes

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Stochastic Process

Often the systems we consider evolve in time and we are interested in their dynamic behaviour, usually involving some randomness.

Examples are

- the length of a queue
- the temperature outside
- the number of students passing Statistics I at ISEG along time
- the number of claims in a portfolio along time

Some definitions

Definition: stochastic process

Given a probability space (Ω, \mathcal{F}, P) and an arbitrary set T , a **stochastic process** is a real and finite function $X(t, \omega)$ defined in $T \times \Omega$, and that for each fixed t $X(t, \omega)$ is a measurable function of $\omega \in \Omega$

$$\{X(t, \omega) : t \in T\}$$

Thus

- A stochastic process $\{X_t\}_{t \in T}$ is a family of random variables $X_t : \Omega \rightarrow \mathbb{R}$ indexed by a parameter t (usually the time).
- For each fixed t , X_t is a random variable

Common notation:

$$\{X(t) ; t \in T\} , \{X(t)\}_{t \in T} , \{X_t ; t \in T\} , \{X_t\}_{t \in T} .$$

$$\{X(t) ; t \geq 0\} , \{X(t)\}_{t \geq 0} , \{X_t ; t = 0, 1, \dots\} , \{X_t\}_{t=0}^{\infty} .$$

Some definitions

Definition: parameter space

T is called the parameter space.

Definition: state space

The set of all possible values of X_t is the state space S .

Definition: trajectory or sample path

Joint realization of the random variables X_t for all $t \in T$. It is a function from T to S .

Examples

- sample path of the number of claims
- sample path of the aggregate claim amount

Specification

Joint distribution function

For an arbitrary finite set t_1, t_2, \dots, t_n of values $t \in T$, the corresponding r.v.'s $X(t_1), X(t_2), \dots, X(t_n)$ have joint distribution function

$$F(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) = P(X(t_1) \leq x_1, \dots, X(t_n) \leq x_n) \quad (1)$$

Definition: temporal law

The temporal law $\{X(t); t \in T\}$ is a family of functions (1) for $n = 1, 2, \dots$ and all possible values $t_j, j = 1, 2, \dots, n$

Alternatively, a stochastic process can be specified by the joint characteristic function given by

$$\varphi_{X(t_1), X(t_2), \dots, X(t_n)}(s_1, s_2, \dots, s_n) = E \left[\exp \left(i \sum_{k=1}^n s_k X(t_k) \right) \right]$$

Classification

Classification of a stochastic process

The classification of a stochastic process can be based on:

- the parametric space T
- the state space S
- the dependence relations between the random variables $X(t)$

Definition

- If T is countable, we say $\{X(t)\}_{t \in T}$ is a **discrete-time** stochastic process
- if T is a continuum, we say $\{X(t)\}_{t \in T}$ is a **continuous-time** stochastic process

Definition

- If S is countable, we say $\{X(t)\}_{t \in T}$ is a **discrete** stochastic process
- if S is a continuum, we say $\{X(t)\}_{t \in T}$ is a **continuous** stochastic process

Classification

Definition: independent increments

A continuous-time stochastic process $\{X(t) : t \in T\}$ is said to have **independent increments** if, for all $t_0 < t_1 < \dots < t_n < t$, the random variables

$$X(t_1) - X(t_0), X(t_2) - X(t_1), \dots, X(t_n) - X(t_{n-1})$$

are independent.

Independent increments

For a process with independent increments, the probability law of $X(t)$ and $X(t) - X(s)$, for all t and s , specifies the process.

Classification

Definition: stationary increments

The stochastic process $\{X(t) : t \in T\}$ has **stationary increments** if the random variable

$$X(t_2 + h) - X(t_1 + h)$$

has the same distribution of

$$X(t_2) - X(t_1)$$

for all t_1, t_2 and $h \leq 0$.

Independent and stationary increments

If a process has independent and stationary increments, the probability law of $X(t)$ specifies the process.

Classification

Definition: counting process

A counting process is a stochastic process, in discrete or continuous time, whose state space is

$$S = \{0, 1, 2, \dots\}$$

with the property that $X(t)$ is a non-decreasing function of t .

Classification

Definition: random walk

Stochastic process such that

- the initial value is X_0
- the process is observed at $t = 1, 2, \dots$
- at time $t = 1$ the process jumps to $X_1 = X_0 + Z_1$, where the size of the jump Z_1 is a r.v. with a given distribution
- at time $t = 2$ the process jumps to $X_2 = X_1 + Z_2$, with Z_2 independent of Z_1 , but with the same distribution

After t jumps

$$X_t = X_0 + Z_1 + \dots + Z_t = X_{t-1} + Z_t$$

where Z_t , $t = 1, 2, \dots$ is a sequence of i.i.d. random variables.

Definition: simple random walk

Random walk where the r.v.'s Z_t take only the values $-1, 0$ and 1 with probabilities p , $1 - p - q$ and q .

Definition: symmetric random walk

Random walk where the r.v.'s Z_t take only the values -1 and 1 with probabilities $1/2$.

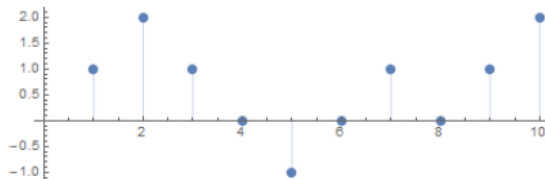
Classification

Definition: white noise

A white noise is a stochastic process that consists of a set of independent and identically distributed random variables. The random variables can be either discrete or continuous and the time set can be either discrete or continuous.

Example: white noise

$\{Z_t : t = 1, 2, \dots\}$ in the random walk is a white noise.



Classification

Definition: stationary stochastic process

A stochastic process $\{X(t) : t \in T\}$ is said to be stationary or strictly stationary, if the joint distribution of

$$(X(t_1), X(t_2), \dots, X(t_n))$$

and

$$(X(t_1 + h), X(t_2 + h), \dots, X(t_n + h))$$

are identical for all h and for all $t_1, t_2, \dots, t_n \in T$, and for all integer n .

Classification

Markov property

The future, given the present, does not depend on the past.

Definition: Markov process

Stochastic process satisfying the Markov property:

$$P(a < X(t_{n+1}) \leq b | X(t_1) = x_1, X(t_2) = x_2, \dots, X(t_n) = x_n) = P(a < X(t_{n+1}) \leq b | X(t_n) = x_n)$$

for all $t_1 < t_2 < \dots < t_n < t_{n+1}$, x_1, \dots, x_n , a and b .

Or, in continuous time,

$$P(a < X(t+h) \leq b | \mathcal{F}_t) = P(a < X(t+h) \leq b | X_t)$$

where \mathcal{F}_t is a natural filtration of the process (σ -algebra generated by the process up to time t).

Markov processes

- A stochastic process with independent increments is a Markov process.
- A random walk is a Markov process.

Classification

Definition: Poisson process

Counting process, with $X(0) = 0$, with independent and stationary increments and such that

$$P(X(t) = k) = \frac{e^{-\lambda t} (\lambda t)^k}{k!}, \quad k = 0, 1, 2, \dots$$

where $\lambda > 0$ is denoted the intensity of the process.

Poisson process

- The Poisson process is a Markov process.