Probability and Stochastic Processes

Master in Actuarial Sciences

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2019/2020

Discrete Time Markov Chains

Discrete Time Markov Chains

Definition: Markov chain

$\Rightarrow$ A **Markov chain** is a Markov process with discrete state space.

Definition: discrete-time Markov chain

$\Rightarrow$ A **Markov chain** is a Markov process with discrete state space.

Consider a (discrete-time) stochastic process $\{X_n : n = 0, 1, 2, \ldots\}$, taking on a finite or countable number of possible values (discrete stochastic process).

- Unless otherwise mentioned, we will assume that $S = \{0, 1, 2, \ldots\}$
- If $X_n = i$, the process is said to be in state i at time n

If we have that

$$P(X_{n+1} = j | X_0 = i_0, X_1 = i_1, \ldots, X_n = i) = P(X_{n+1} = j)|X_n = i) = P_{ij}^{n,n+1}$$

then the SP is called **Markov chain**.

Discrete Time Markov Chains

Definition: homogeneous Markov chain

When the transition probabilities are independent of time (stationary transition probabilities):

$$P_{ij}^{n,n+1} = P_{ij}^{(1)} = P_{ij}, \qquad P_{ij}^{n,n+s} = P_{ij}^{(s)} \qquad \forall s = 1, 2, \dots$$

One-step transition probabilities

Matrix of the one-step transition probabilities

The one-step transition probabilities can be represented by a matrix $\mathbf{P} = [P_{ij}]$:

$$\mathbf{P} = \begin{pmatrix} P_{00} & P_{01} & P_{02} & \dots \\ P_{10} & P_{11} & P_{12} & \dots \\ P_{20} & P_{21} & P_{22} & \dots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

- Line $i+1$ of $\mathbf{P}$ represents the probability distribution of X_{n+1} when $X_n = i$:

$$P_{ij} \geqslant 0, \qquad i,j = 0,1,2,\dots$$
$$\sum_{j=0}^{\infty} P_{ij} = 1, \qquad i = 0,1,2,\dots$$

The probability of a path

Specification

If we know

- $\mathbf{P}$
- $p_i = P(X_0 = i)$, $i = 0, 1, 2, \ldots$: the initial distribution

then the process is completely specified:

$$\begin{aligned}
&P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) = \\
&\quad = \quad P(X_n = i_n | X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}) P(X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}) \\
&\quad = \quad P(X_n = i_n | X_{n-1} = i_{n-1}) P(X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}) \\
&\qquad\qquad \vdots \\
&\quad = \quad P_{i_{n-1}, i_n} P_{i_{n-2}, i_{n-1}} \ldots P_{i_0, i_1} p_{i_0}
\end{aligned}$$

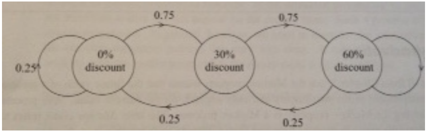
Example

No claim discount (NCD)

A company offers discounts of 0%, 30% and 60% of the full premium. A policyholder's status is determined by the following rules:

- A new policyholder starts at the 0% level;
- If no claim is made during the current year then he moves up one discount level or remains at the 60% level;
- If one or more claims are made he moves down one level, or remains at the 0% discount.

Assuming that the probability of no claim per year is 3/4, determine **P**.



n-step transition probabilities and the Chapman-Kolmogorov equations

n-step transition probabilities

$$P_{ij}^{(n)} = P(X_{s+n} = j | X_s = i)$$

Chapman-Kolmogorov equations

$$P_{ij}^{(m+n)} = \sum_{k=0}^{\infty} P_{ik}^{(m)} p_{kj}^{(n)}$$

for all non-negative integers m and n with

$$P_{ij}^{(0)} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}, \qquad \text{i.e.} \qquad \mathbf{P}^{(0)} = \mathbf{I}$$

Thus

$$\mathbf{P}^{(m+n)} = \mathbf{P}^{(m)}\mathbf{P}^{(n)}$$

and hence

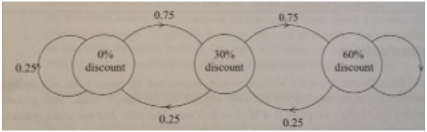
$$\mathbf{P}^{(n)} = \mathbf{P}\mathbf{P}^{(n-1)} = \mathbf{P}\mathbf{P}\mathbf{P}^{(n-2)} = \cdots = \mathbf{P}^{n}$$

Example

No claim discount (NCD)

Determine, for the NCD model considered,

- The probability that the policyholder has a discount of 30% after being in the company for three years.
- The probability that a policyholder that is now with 30% discount ends up 4 years later at the same level.



Example

Example

Consider a Markov chain with the following probability distribution at time $n = 0$ and one-step transition probability matrix:

k	$P(X_0 = k)$
0	1/2
1	1/2
2	0
3	0

$$\mathbf{P} = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \end{bmatrix}$$

Compute $P(X_2 = 3)$.

First step analysis

First step analysis

Quite a number of functionals on a Markov chain can be evaluated by a technique called first step analysis.

Example: first step analysis

Consider a Markov chain with $S = \{0, 1, 2\}$ such that

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ \alpha & \beta & \gamma \\ 0 & 0 & 1 \end{bmatrix}$$

with $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = 1$

- In which of the states is the chain absorbed?
- How long does it take to reach one of the absorbing states?

First step analysis

Example: first step analysis

Let

$$T = \min\{n \geqslant 0 : X_n = 0 \text{ or } X_n = 2\}$$

be the time of absorption of the process.

- Probability that the chain is absorbed by state 0

$$u = P(X_T = 0 | X_0 = 1)$$

- Expected time to absorption

$$v = E[T | X_0 = 1]$$

After the first step the process is in one of the states and

$$P(X_T = 0 | X_1 = 0) = 1$$
$$P(X_T = 0 | X_1 = 1) = u$$
$$P(X_T = 0 | X_1 = 2) = 0$$

First step analysis

Example: first step analysis

Using the theorem of the total probability

$$\begin{aligned} u &= P(X_T = 0 | X_0 = 1) \\ &= \sum_{k=0}^{2} P(X_T = 0 | X_0 = 1, X_1 = k)\, P(X_1 = k | X_0 = 1) \\ &= \sum_{k=0}^{2} P(X_T = 0 | X_1 = k)\, P(X_1 = k | X_0 = 1) \\ &= \alpha + u\beta \end{aligned}$$

Hence

$$u = \frac{\alpha}{1-\beta} = \frac{\alpha}{\alpha+\gamma}$$

Show that

$$v = \frac{1}{1-\beta}$$

First step analysis

First step analysis

Let a finite Markov chain $\{X_n\}$ with states $n = 0, 1, \ldots, N$, numbered in such way that the first r states are transient and the remaining $r, \ldots, N$ are absorbing.
Then **P** may be written as

$$\mathbf{P} = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix} \qquad \text{where} \quad Q_{ij} = P_{ij} \text{ for } 0 \leqslant i, j < r$$

First step analysis

First step analysis

Let U_{ik} be the probability of absorption in k when the process starts at i ($r \leqslant k \leqslant N$ and $0 \leqslant i < r$):

$$\begin{aligned} U_{ik} &= P(\text{absorption in } k | X_0 = i) \\ &= \sum_{j=0}^{N} P(\text{absorption in } k | X_0 = i, X_1 = j) P_{ij} \\ &= P_{ik} + \sum_{j=r, j \neq k}^{N} (P_{ij} \times 0) + \sum_{j=0}^{r-1} (P_{ij}\, U_{jk}) \end{aligned}$$

Thus

$$U_{ik} = P_{ik} + \sum_{j=0}^{r-1} P_{ij}\, U_{jk}, \qquad \text{for} \quad i = 0, 1, \ldots, r-1$$

First step analysis

First step analysis

Let T be the time of absorption:

$$T = \min\{n \geqslant 0 : X_n \geqslant r\}$$

and suppose that there is a rate $g(i)$, $i = 0, \ldots, r-1$, associated to the transient states. Let w_i be the mean rate until absorption if the process starts at i:

$$w_i = E\left[\sum_{n=0}^{T-1} g(X_n) | X_0 = i\right]$$

- The choice $g(i) = 1$, for all i, gives v_i, the mean time until absorption starting from i.
- The choice
$$g(i) = \delta_{ik} = \begin{cases} 1, & \text{if } i = k \\ 0, & \text{if } i \neq k \end{cases}$$
gives w_{ik}, the mean number of visits to state k before absorption.

First step analysis

First step analysis

We have (from the total probability theorem)

$$w_i = g(i) + \sum_{j=0}^{r-1} w_j P_{ij}, \qquad \text{for} \quad i = 0, 1, \ldots, r-1$$

If $g(i) = 1$ for all i, we obtain

$$v_i = 1 + \sum_{j=0}^{r-1} v_j P_{ij}, \qquad \text{for} \quad i = 0, 1, \ldots, r-1$$

If $g(i) = \delta_{ik}$, we obtain

$$w_{ik} = \delta_{ik} + \sum_{j=0}^{r-1} w_{jk} P_{ij}, \qquad \text{for} \quad i = 0, 1, \ldots, r-1$$

First step analysis

Example

Consider the following Markov chain:

$$P = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.1 & 0.2 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.6 & 0.1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Consider that the chain starts at state 1.

- What is the probability that the chain is absorbed in state 0?
- What is the mean number of steps until abosorption?
- What is the mean number of visits to state 2 before absorption?

First step analysis

Example

Consider a NCD model as follows: there are four levels of discount: no discount, 25% discount; 40% discount and 60% discount. The rules for moving up the discount scale are as in the previous example, but in the case of a claim during the current year, the discount status moves down one or two steps according to whether or not the previous-year was claim free.

1) Describe the model as a Markov chain, by defining the states and the one-step transition probability matrix.
2) If you are at 0% discount, what is the probability that you are on the maximum discount after 5 years.
3) Calculate the mean number of years that a policyholder has a 25% discount before he reaches the maximum discount.
4) Determine the mean number of years that a policyholder just entering the company takes to get, for the first time, to the maximum discount level.

Classification of states

Definition

State j is **accessible** ($i \to j$) from state i if $P_{ij}^{(n)} > 0$, for some $n > 0$.

Definition

States i and j **communicate** ($i \longleftrightarrow j$) if i is accessible from j and j is accessible from i:

$$\exists n, m : \; P_{ij}^{(n)} > 0 \quad \text{and} \quad P_{ji}^{(m)} > 0$$

Se não , ou $P_{ij}^{(n)} = 0, \forall n$, ou $P_{ji}^{(n)} = 0, \forall n$.

Theorem

The concept of communication is an equivalence relation: (reflexivity; symmetry and transitivity).

- Equivalence classes

Classification of states

Example

Consider the Markov chain

$$A = \begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix} \end{array}$$

Make a partition of the state space in equivalent classes.

Classification of states

Example

Consider the Markov chain

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Make a partition of the state space in equivalent classes.

Classification of states

Example

Let $S = \{0, 1, 2, \ldots, a\}$ and

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ a-1 \\ a \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & \cdots & a-2 & a-1 & a \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ q & 0 & p & 0 & \cdots & 0 & 0 & 0 \\ 0 & q & 0 & p & \cdots & 0 & 0 & 0 \\ & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & q & 0 & p \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Make a partition of the state space in equivalent classes.

Classification of states

Definition: irreducible Markov chain

A Markov chain is said to be irreducible if the equivalence relation ($\longleftrightarrow$) induces only one class.

Definition: period of a state

The period of state i, $d(i)$ is the greatest common divisor of all integer $n \geqslant 1$ such that $P_{ii}^{(n)} > 0$:

$$d(i) = \gcd\{n \geqslant 1 : P_{ii}^{(n)} > 0\}$$

If $P_{ii}^{(n)} = 0$, for all $n \geqslant 1$, we define $d(i) = 0$.

Definition: aperiodic Markov chain

A Markov chain in which each state has period 1 is called aperiodic.

Theorem

Two communicating states have the same period:

$$i \longleftrightarrow j \Longrightarrow d(i) = d(j)$$

Classification of states

Theorem

If state i has period $d(i)$, then there exists an integer N dependent of i, such that for all $n \geqslant N$ we have

$$P_{ii}^{(n\,d(i))} > 0$$

i.e. the return to state i can occur at all the sufficiently large multiples of $d(i)$.

Classification of states

$f_{ii}^{(n)}$

For an arbitrary state i, consider

$$f_{ii}^{(n)} = P(X_n = i \text{ and } X_k \neq i, k = 1, 2, \ldots, n-1 | X_0 = i)$$

- $f_{ii}^{(n)}$ is the probability that the first return to state i occurs in n steps.
- $f_{ii}^{(n)}$ can be computed recursively

$$P_{ii}^{(n)} = \sum_{k=0}^{n} f_{ii}^{(k)} P_{ii}^{(n-k)}, \quad \forall n \geqslant 1$$

with (assuming) $f_{ii}^{(0)} = 0$, for all i.

Classification of states

$f_{ij}^{(n)}$

For an arbitrary state i, consider

$$f_{ij}^{(n)} = P(X_n = j \text{ and } X_k \neq j, k = 1, 2, \ldots, n-1 | X_0 = i)$$

- $f_{ij}^{(n)}$ is the probability that the first return to state j, coming from state i, occurs in n steps.
- $f_{ij}^{(n)}$ can be computed recursively

$$P_{ij}^{(n)} = \sum_{k=0}^{n} f_{ij}^{(k)} P_{jj}^{(n-k)}, \quad i \neq j, \quad \forall n \geqslant 1$$

Classification of states

Probability of ever return to state i

$$f_{ii} = \sum_{n=0}^{\infty} f_{ii}^{(n)}$$

Definition: recurrent and transient state

- State i is **recurrent** if $f_{ii} = 1$.
- State i is **transient** when $f_{ii} < 1$.

Theorem

State i is recurrent if and only if $\sum_{n=0}^{\infty} P_{ii}^{(n)} = \infty$

Coroliary

If $i \longleftrightarrow j$ and i is recurrent, then j is recurrent.

Classification of states

- If i is transient, $f_{ii}^{(n)}$ is not a proper distribution.

Definition: expected number of transitions needed to return to state i

$$m_i = \begin{cases} \sum_{n=1}^{+\infty} n\, f_{ii}^{(n)}, & \text{if } i \text{ is recurrent} \\ +\infty, & \text{if } i \text{ is transient} \end{cases}$$

Definition: null recurrent and positive recurrent

- State i is **null recurrent** if it is recurrent and $m_i = \infty$.
- State i is **positive recurrent** if it is recurrent and $m_i < \infty$.

Definition: ergodic state

A state is called **ergodic** if it is positive recurrent and aperiodic.

Classification of states

Definition: closed class

A set C of states is a closed class if

$$P_{ij} = 0, \quad \forall i \in C, j \notin C$$

Definition: irreducible class

A set C of states is an irreducible class if $i \longleftrightarrow j$, for all $i, j \in C$.

Definition: absorbing state

State i is absorbing if $P_{ii} = 1$.

Definition

An irreducible class C is aperiodic (or recurrent, or null recurrent, ...) if all the states in C have that property.

Classification of states

Theorem

If S is finite, then

- At least one state is recurrent
- All recurrent states are positive recurrent

Theorem

- The recurrent classes are closed.
- In a finite chain all the closed irreducible class is recurrent.

Theorem: decomposition

Given a state space S of a Markov chain, there exists a unique partition of S

$$S = T \cup C_1 \cup C_2 \cup \dots$$

where T is the class of transient states and C's are closed, irreducible classes of recurrent states

Stationary probability distribution

Stationary probability distribution

$\pi = [\pi_0, \pi_1, \ldots]$ is a **stationary probability distribution** for the Markov chain with transition probability matrix $\mathbf{P}$ if the following conditions hold:

$$\pi = \pi \mathbf{P} \qquad \text{and} \qquad \sum_{k \in S} \pi_k = 1 \qquad \pi_k \geqslant 0, k \in S$$

- π is the left eigenvector corresponding to the eigenvalue 1 (or the eigenvector corresponding to the eigenvalue 1 of $\mathbf{P}^T$), normalised in such way that it is a probability distribution.

Limit distribution

Remark

The theory of Markov chains turns out to be very simple when the number of states is finite. In that case, if the chain is irreducible than all states are positive recurrent.

Definition: regular Markov chain

A probability matrix $\mathbf{P}$, with states $0, 1, ..., N$, or the corresponding Markov chain, is said to be regular if there exists a k such that all the elements of the matrix $\mathbf{P}^n$, $n \geqslant k$ are strictly positive.

- A markov chain is regular if at some point in time all elements of the transition probability matrix are positive.

Theorem

A finite, irreducible and aperiodic transition probability matrix $\mathbf{P}$ is regular.

Limit distribution

Definition: limiting distribution

The limiting distribution $\pi = (\pi_0, \pi_1, ..., \pi_N)$ is given by

$$\pi_j = \lim_{n \to \infty} P_{ij}^{(n)} > 0.$$

Example

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$P^{100} \approx \begin{bmatrix} 0.33333 & 0.33333 & 0.33333 \\ 0.33333 & 0.33333 & 0.33333 \\ 0.33333 & 0.33333 & 0.33333 \end{bmatrix}$$

Limit distribution

Example

Consider the Markov chain with the following transition matrix:

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0.9 & 0.1 & 0 & 0 & 0 & 0 & 0 \\ 0.9 & 0 & 0.1 & 0 & 0 & 0 & 0 \\ 0.9 & 0 & 0 & 0.1 & 0 & 0 & 0 \\ 0.9 & 0 & 0 & 0 & 0.1 & 0 & 0 \\ 0.9 & 0 & 0 & 0 & 0 & 0.1 & 0 \\ 0.9 & 0 & 0 & 0 & 0 & 0 & 0.1 \\ 0.9 & 0 & 0 & 0 & 0 & 0 & 0.1 \end{pmatrix} \end{array}$$

Long run:

$$P^8 = \begin{bmatrix} .9 & .09 & .009 & .0009 & .00009 & 9.0\times10^{-6} & 1.0\times10^{-6} \\ .9 & .09 & .009 & .0009 & .00009 & 9.0\times10^{-6} & 1.0\times10^{-6} \\ .9 & .09 & .009 & .0009 & .00009 & 9.0\times10^{-6} & 1.0\times10^{-6} \\ .9 & .09 & .009 & .0009 & .00009 & 9.0\times10^{-6} & 1.0\times10^{-6} \\ .9 & .09 & .009 & .0009 & .00009 & 9.0\times10^{-6} & 1.0\times10^{-6} \\ .9 & .09 & .009 & .0009 & .00009 & 9.0\times10^{-6} & 1.0\times10^{-6} \\ .9 & .09 & .009 & .0009 & .00009 & 9.0\times10^{-6} & 1.0\times10^{-6} \end{bmatrix}$$

Stationary and limit distributions

Example

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \to \mathbf{P}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \mathbf{P}^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdots$$

P does not have limiting distribution:

$$P_{ii}^{(n)} = \begin{cases} 1 & i = 1,2 \quad n \text{ even} \\ 0 & i = 1,2 \quad n \text{ odd} \end{cases} \Rightarrow \nexists \text{ dist.lim.}$$

(P is not regular)
But P has stationary distribution:

$$\underbrace{\begin{bmatrix} 1/2 & 1/2 \end{bmatrix}}_{\pi} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{P} = \underbrace{\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}}_{\pi}$$

Limit distribution

Theorem: regular chains

A Markov chain with a regular transition probability matrix has a limiting distribution, which is independent of the initial state and is given by the only positive solution to

$$\pi = \pi \mathbf{P}, \qquad \sum_{k=0}^{N} \pi_k = 1$$

Remarks

- π_j is the probability of finding the process in state j, when the process has been in operation for a long time (independent of the initial state):

$$P^n \longrightarrow \underbrace{\begin{pmatrix} \pi_1 & \pi_2 & \cdots & \pi_N \\ \pi_1 & \pi_2 & \cdots & \pi_N \\ \vdots & \vdots & & \vdots \end{pmatrix}}_{\text{all rows are equal to } \pi} \qquad \text{as } n \to +\infty$$

Limit distribution

Remarks

- π_j also represents the long run fraction of time that the process $\{X_n\}$ spends in state j.
- For a recurrent, irreducible and aperiodic Markov chain we have

$$\lim_{n\to\infty} P_{ij}^{(n)} = \pi_j = \frac{1}{m_j}, \quad \forall i,j = 1,\dots,N$$

(m_j is the expected number of steps to return to state j)

Limit distribution

Example

Consider

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

P is regular and computationally we can show that

$$P^{100} \approx \begin{bmatrix} 0.33333 & 0.33333 & 0.33333 \\ 0.33333 & 0.33333 & 0.33333 \\ 0.33333 & 0.33333 & 0.33333 \end{bmatrix}$$

Show that the limit is actually

$$\pi = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix}$$

solving equation

$$\pi P = \pi \iff \begin{cases} 1/2\pi_2 + 1/2\pi_3 & = & \pi_1 \\ 1/2\pi_1 + 1/2\pi_3 & = & \pi_2 \\ 1/2\pi_1 + 1/2\pi_2 & = & \pi_3 \end{cases}$$