Probability and Stochastic Processes

Master in Actuarial Sciences

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Continuous time homogeneous Markov chains

Continuous time homogeneous Markov chains

Definition: continuous time homogeneous Markov chain

Markov process, with countable state space S, in **continuous time**, that has **stationary transition rates**:

- For all $i, j \in S$ exists a probability function $p_{ij}(t)$ such that

$$P(X(s+t) = j | X(s) = i) = p_{ij}(t), \qquad \text{for all } s, t > 0$$

  independent of s.

Remark

For a time homogeneous Markov process $\{X(t)\}_{t \geqslant 0}$, given its evolution up to any "current" time s, the probabilistic description of its behavior at all future times depends only on the current state $X(s) = i$, and not on the previous history of the process nor on the time s itself.

The transition probability matrix

Definition: transition probability matrix at time t

For each $t \geqslant 0$, we define the matrix $\mathbf{P}(t) = \left[p_{ij}(t)\right]_{i,j \in S}$.

Remark

Note that

- For each t, $\mathbf{P}(t)$ is a stochastic matrix:

$$p_{ij}(t) \geqslant 0 \ \ \forall i,j \in S \quad \text{and} \quad \sum_{j \in S} p_{ij}(t) = 1 \ \ \forall i \in S$$

- $p_{ij}(0) = 1$, if $i = j$ and $p_{ij}(0) = 0$, if $i \neq j$:

$$p_{ij}(0) = \delta_{ij} \Longleftrightarrow \mathbf{P}(0) = \mathbf{I}$$

Chapman-Kolmogorov equations

Chapman-Kolmogorov equations

$$p_{ij}(s+t) = \sum_{k \in S} p_{ik}(s) p_{kj}(t), \quad t, s \geqslant 0, \qquad \forall i, j \in S$$

The transition matrix **P** becomes

$$\left[p_{ij}(t+s)\right]_{i,j \in S} = \mathbf{P}(t+s) = \mathbf{P}(s)\mathbf{P}(t)$$

The matrix of transition rates

- We will assume that the probability functions $p_{ij}(t)$ are differentiable.

Definition: transition rates

The transition rate, intensity rate or force of transition, from i to j is defined by

$$q_{ij} = p'_{ij}(0) \quad \forall i, j \in S$$

Transition rates

Then, for all $t, h \geqslant 0$,

$$\begin{aligned} P(X(t+h)=j|X(t)=i) &= p_{ij}(h) && \text{(homogeneous process)} \\ &= p_{ij}(0) + q_{ij}h + o(h), \quad \text{as } h \to 0 && \text{(1st order approximation)} \\ &= \delta_{ij} + q_{ij}h + o(h), \quad \text{as } h \to 0 \end{aligned}$$

that is, q_{ij} is the (instataneous) transition rate of the process from state i to state j, and

$$p_{ij}(h) = \begin{cases} 1 + q_{ij}h + o(h), & i = j \\ q_{ij}h + o(h), & i \neq j \end{cases}$$

The matrix of transition rates

Remark

Since $\sum_{j\in S} p_{ij}(h) = 1$ e $p_{ii}(h) = 1 + q_{ii}h$, then

$$q_{ii} = -\sum_{j\neq i} q_{ij} \qquad \text{and} \qquad \sum_{j\in S} q_{ij} = 0$$

Definition: Matrix of transition rates

The matrix

$$\mathbf{Q} = \left[q_{ij}\right]_{i,j\in S}$$

is the transition rate matrix, or intensity matrix, or generator, of the process.

- We assume that:

$$\sum_{j\in S} q_{ij} = 0, \quad \forall i$$

$$0 \leqslant q_{ij} < \infty, \quad \forall i \neq j$$

$$0 \leqslant -q_{ii} < \infty, \quad \forall i$$

The matrix of transition rates

Remarks

- q_{ij} **are not probablities**
- It is possible to build the transition probability matrix from the matrix of transition rates
- The matrix of transition rates specifies the probability law of the process

The matrix of transition rates

Example

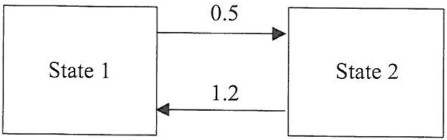
Consider the time continuous Poisson process, such that

$$p_{ij}(t) = \frac{e^{-\lambda t}(\lambda t)^{(j-i)}}{(j-i)!}$$

Build the generator matrix of the process.

Example

Consider the Markov process with 2 states and the following transition rates.



$$\begin{pmatrix} -0.5 & 0.5 \\ 1.2 & -1.2 \end{pmatrix}$$

The forward differential equations

Theorem: forward differential equations

$$p'_{ij}(t) = \sum_{k \in S} p_{ik}(t) q_{kj}$$

under the initial conditions $p_{ij}(0) = \delta_{ij}$.
In matrix form:

$$\begin{cases} P'(t) & = & P(t)Q \\ P(0) & = & I \end{cases}$$

Proof

$$p_{ij}(t+h) = \sum_{k \in S} p_{ik}(t) p_{kj}(h) = \cdots = \sum_{k \in S} p_{ik}(t) \left[q_{kj} h + o(h) \right] + p_{ij}(t)$$

The backward differential equations

Theorem: backward differential equations

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$$p'_{ij}(t) = \sum_{k \in S} q_{ik} p_{kj}(t)$$

under the initial conditions $p_{ij}(0) = \delta_{ij}$.
In matrix form:

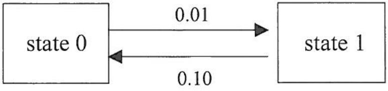
$$\begin{cases} P'(t) & = & QP(t) \\ P(0) & = & I \end{cases}$$

Proof

$$p_{ij}(t+h) = \sum_{k \in S} p_{ik}(h) p_{kj}(t) = \cdots = \sum_{k \in S} \left[q_{ik} h + o(h) \right] p_{kj}(t) + p_{ij}(t)$$

The forward and backward differential equations

Example



$$\frac{d}{dt}P_{01}(t) = P_{00}(t)\mu_{01} + P_{01}(t)\mu_{11} = 0,01P_{00}(t) - 0,10P_{01}(t)$$

The forward and backward differential equations

Example

General two state process, $S = \{0, 1\}$

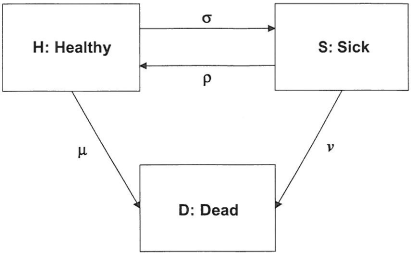
$$\mathbf{Q} = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}, \qquad \lambda, \mu > 0$$

Find $P(t)$.

The forward and backward differential equations

Example

Time homogeneous health-sickness-death model.



Holding times

Holding time

$$T_0 = \inf\{t : X(t) \neq X(0)\}$$

Theorem

The holding time in any state $i \in S$ of a time homogeneous Markov process with transition rate matrix Q is exponential distributed with mean $-\dfrac{1}{q_{ii}}$.

This is to say, that

$$p_{\overline{ii}}(t) = P(T_0 > t | X(0) = i) = e^{q_{ii}t}$$

where

- $p_{\overline{ii}}(t)$ is the probability that the process remains in state i throughout a perid of length t.

Holding times

Notation

- Sometimes we use $q_i = -q_{ii}$, so $p_{\overline{ii}}(t) = e^{-q_i t}$
- $p_{\overline{ii}}(t)$ is the probability that the process will remain in state i during an interval of range t:

$$p_{\overline{ii}}(t) = P(T_0 > t | X(0) = i)$$

- $p_{ii}(t)$ is the probability that, being at state i, after a time interval of length t we are still back at state i:

$$p_{ii}(t) = P(X(t) = i | X(0) = i)$$

- Note that $p_{\overline{ii}}(t) \neq p_{ii}(t)$

Holding times

Holding time at state i

$$p_{\overline{ii}}(t) = P(T_0 > t | X(0) = i) = e^{q_{ii}t} = e^{-q_i t}, \qquad T_i \sim Exp\left(-\frac{1}{q_{ii}}\right)$$

Proof

$$p_{\overline{ii}}(t+h) = p_{\overline{ii}}(t) p_{\overline{ii}}(h)$$

$$p_{\overline{ii}}(h) = 1 - \sum_{j \neq i} p_{ij}(h) \quad \text{and} \quad p_{ij}(h) = hq_{ij} + o(h) \ \text{ for } j \neq i$$

Hence

$$p_{\overline{ii}}(t+h) = p_{\overline{ii}}(t)\left[1 + hq_{ii} + o(h)\right]$$

from where

$$p'_{\overline{ii}}(t) = q_{ii} p_{\overline{ii}}(t)$$

So, given that $p_{\overline{ii}}(0) = 1$, we find

$$p_{\overline{ii}}(t) = e^{q_{ii}t}$$

Holding times

Theorem

The probability that the process goes into state j when it leaves state i is $\dfrac{q_{ij}}{-q_{ii}}$.

Proof

$$\begin{aligned} P(X(t+h)=j|X(t)=i) &= q_{ij}h+o(h), \quad i\neq j \\ P(X(t+h)\neq i|X(t)=i) &= -q_{ii}h+o(h) \end{aligned}$$

Hence

$$\begin{aligned} P(X(t+h)=j|X(t)=i, X(t+h)\neq i) &= \frac{P(X(t+h)=j, j\neq i|X(t)=i)}{P(X(t+h)\neq i|X(t)=i)} \\ &= \frac{q_{ij}h+o(h)}{-q_{ii}h+o(h)} \longrightarrow \frac{q_{ij}}{-q_{ii}} \end{aligned}$$

Holding times

Definition: m_i

- Let m_i be the expected time for a process to reach state k given that it is currently in state i.
- Then, m_i can be computed using a recursive formula

$$m_i = \frac{1}{-q_{ii}} + \sum_{j \neq k,i} \frac{q_{ij}}{-q_{ii}} m_j$$

Jumping chain or embedded Markov chain

Definition: jumping chain

Let

- $\{X(t)\}_{t\geqslant 0}$ be a CTMC
- Q be the matrix of transition rates (jump intensities) of the process
- W_n be the time of the n-th jump:

$$W_0 = 0, \qquad T_n = W_{n+1} - W_n \qquad \text{and} \qquad P(T_n > t | X(W_n) = i) = e^{q_{ii}t}$$

Consider the process

- $X_n^* = X(W_n), n \geqslant 0$

Then

- $\{X_n^*\}_{n\geqslant 0}$ is a Markov chain in discrete time, called the **jumping chain** or the embedded Markov chain.

$$\mathbf{P}^* = (p_{ij}^*)_{i,j\in S}$$

Jumping chain or embedded Markov chain

Jumping chain of a CTMC

$$P^* = \left(p^*_{ij}\right)_{i,j\in S}$$

such that

$$\begin{cases} \begin{cases} p^*_{ij} & = & \dfrac{q_{ij}}{-q_{ii}}, \quad \text{if } j \neq i \\ & & \\ p^*_{ii} & = & 0 \end{cases} & \text{if } q_{ii} < 0 \\ & \\ \begin{cases} p^*_{ij} & = & 0, \quad \text{if } j \neq i \\ p^*_{ii} & = & 1 \end{cases} & \text{if } q_{ii} = 0 \end{cases}$$

Remarks

- Here we are interested in the time instants at which jumps occur
- The transition probability from i to j is the probability that the process jumps from i to j.
- We move from a continuous time (homogeneous) MC to a discrete time (homogeneous) MC

Jumping chain or embedded Markov chain

Reachable state

State j is said to be **reachable** from stacte i for a CTMC if

$$P(X(s) = j|X(0) = i) = p_{ij}(s) > 0, \quad \text{for some } s \geqslant 0$$

Communicating states and classes

- As with discrete-time chains, i and j are said to **communicate** if state j is reachable from state i, and state i is reachable from state j
- It is immediate that i and j communicate in continuous time if and only if they do so for the embedded discrete-time chain $\{X_n^*\}_{n\geqslant 0}$, *i.e.* they communicate in continuous-time if and only if they do so at transition epochs
- Thus, once again, we can partition the state space into disjoint communication classes

$$S = C_1 \cup C_2 \cup C_3 \cup \cdots$$

- An **irreducible chain** is a chain for which all states communicate ($S = C_1$, one communication class)
- A CTMC is irreducible if and only if its embedded chain is irreducible
- $\mathbf{Q}$ is said to be irreducible if $\mathbf{P}^*$ is irreducible

Jumping chain or embedded Markov chain

Recurrent state

- State i is called **recurrent** if the chain will re-visit state i with certainty (with probability 1)
- Otherwise, the state is **transient**
- State i is is recurrent/transient for a CTMC if and only if it is recurrent/transient for the embedded discretre-time chain (although positive recurrent/null recurrent may be different in the two chains)

Periodicity of states

There are no problems of periodicity for the CTMC, although there may exist for $\{X_n^*\}_{n \geqslant 0}$

Jumping chain or embedded Markov chain

Example

Consider the Markov chain in continuous time with the following generator matrix:

$$Q = \begin{pmatrix} -4 & 1 & 0 & 0 & 3 \\ 2 & -6 & 3 & 1 & 0 \\ 0 & 0 & -5 & 5 & 0 \\ 0 & 0 & 4 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

1. Classify the states.
2. Calculate the probability that the process is absorbed at state 5 if it starts from state 1.

Stationary and limiting distribution for a single closed class

Definition

Consider an irreducible CTMC. A probability distribution in S

$$\pi_j \geqslant 0, \quad \forall j \in S \qquad \text{and} \qquad \sum_{j \in S} \pi_j = 1$$

is **stationary** if and only if, for all $t > 0$

$$\boldsymbol{\pi} \mathbf{P}(t) = \boldsymbol{\pi}$$

- It is also called **equilibrium distribution** and it gives the long term proportion of time spent in each state.
- If $\boldsymbol{\pi}$ is stationary, then $\pi_j > 0$, for all j

Stationary and limiting distribution for a single closed class

Theorem

A distribution $\boldsymbol{\pi}$ is stationary if and only if

$$\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}$$

Theorem

Let S be the a single closed class (irreducible chain) of a CTMC. Then, one of the follwoing two is true

1) There is a unique stationary distribution $\boldsymbol{\pi}$ and

$$\lim_{t\to\infty} p_{ij}(t) = \pi_j, \quad \forall i, j \in S$$

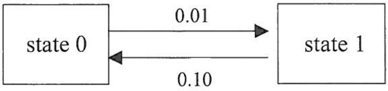
(if S is finite, this case necessarily holds)

2) There is no stationary distribution and

$$\lim_{t\to\infty} p_{ij}(t) = 0, \quad \forall i, j \in S$$

Stationary and limiting distribution for a single closed class

Example



Find the equilibrium distribution of the chain.

Exercise

Vehicles in a certain country are required to be assessed every year for road-worthiness. At one vehicle assessment centre, drivers wait for an average of 15 minutes before the road-worthiness assessment of their vehicle commences. The assessment takes on average 20 minutes to complete. Following the assessment, 80% of vehicles are passed as road-worthy allowing the driver to drive home. A further 15% of vehicles are categorised as "minor fail", these vehicles require on average 30 minuutes of repair work before the driver is allowed to drive home. The remaining 5% of vehicles are categorised as "significant fail", these vehicles require on average three hours of repair work before the driver can go home.
A continuous-time Markov model is to be used to model the operation of the vehicle assessment centre, with states W (waiting for the assessment), A (assessment taking place), M (minor repair taking place), S (significant repair taking palce) and H (travelling home).

- Explain what assumption must be made about the distribution of the time spent in each state
- Write down the generator matrix of this process.
- Use Kolmogorov's Forward Equations to:
  - Write down differential equations satisfied by $p_{WM}(t)$ and by $p_{WA}(t)$
  - Verify that $p_{WA}(t) = 4e^{-t/20} - 4e^{-t/15}$, for $t \geqslant 0$, where t is measured in minutes.
  - Derive an expression for $p_{WM}(t)$, for $t \geqslant 0$.
- Let T_i be the expected length of time (in minutes) until the vehicle can be driven home given that the assessment process is currently in state i.
  - Explain why $T_W = 15 + T_A$.
  - Derive corresponding equations for T_A, T_M and T_S.
  - Calculate T_W.