Master in Actuarial Sciences

Probability and Stochastic Processes

07/01/2020

Time allowed: Three hours

Instructions:

1. This paper contains 6 questions and comprises 3 pages including the title page.
2. Enter all requested details on the cover sheet.
3. You have 10 minutes of reading time. You must not start writing your answers until instructed to do so.
4. Number the pages of the paper where you are going to write your answers.
5. Attempt all 6 questions.
6. Begin your answer to each of the 6 questions on a new page.
7. Marks are shown in brackets. Total marks: 200.
8. Show calculations where appropriate.
9. An approved calculator may be used.
10. The distributed Formulary and the Formulae and Tables for Actuarial Examinations (the 2002 edition) may be used. **Note that, unless otherwise stated, the parametrisation used for the different distributions is that of the distributed Formulary.**

1. Claim sizes in a certain line of business are modelled by a log-normal distribution with $\mu = 8.2$ and $\sigma = 2.1$. The insurer defines a claim to be a large claim if the claim size exceeds 40 000.

(a) Calculate the probabilities that the size of a claim exceeds 40 000 and 42 000. [05]

(b) Calculate the probability that the size of a large claim (as defined by the insurer) exceeds 42 000. [10]

(c) Calculate the probability that a random sample of 7 claims includes exactly 3 large claims, that is, three claims which exceed 40 000 and four which are less than 40 000. [10]

(d) From a sample of 10 claims, what is the probability that the maximum claim exceeds 42 000? [05]

2. Let $X_1 \sim Gamma(\alpha = 2, \theta = 3)$ and $X_2 \sim Exp(\theta = 3)$ be independent random variables. To model the losses on a certain line of business, an actuary considers two different models. First she considers losses, Y, to be a mixture of X_1 and X_2 with weights 3/4 and 1/4, respectively. Afterwards she considers that losses are $Z = \frac{3}{4}X_1 + \frac{1}{4}X_2$.

(a) Do variables Y and Z have the same expected value? Justify your answer. [10]

(b) Determine the variances of Y and Z and compare them. [10]

(c) Knowing that $F_1(x) = 1 - e^{-\frac{x}{3}} - \frac{x}{3}e^{-\frac{x}{3}}$, for $x \geqslant 0$, is the distribution function of X_1, obtain the hazard rate of Y. Classify the tail of this random variable according to this criteria. [10]

(d) Obtain the 99,5% percentile of X_2, $q_{0.995}$, and compute the probability that Y does not exceed that value $q_{0.995}$. Comment on the results. [10]

(e) The actuary considers yet another model, W, where losses are described by a two component spliced distribution in which the density function is proportional to that of X_1 in the interval $(0, 5)$ and it is proportional to that of X_2 for values above 5. Write the density function of this model, ensuring that the resulting density is continuous. [10]

3. Let X and Y be random variables with a joint distribution function given by

$$H(x, y) = \left[1 + e^{-x} + e^{-y}\right]^{-1}$$

(a) Show that X and Y have standard (univariate) logistic distributions, *i.e* $F_X(x) = (1 + e^{-x})^{-1}$ and $F_Y(y) = (1 + e^{-y})^{-1}$, and that the copula of X and Y is given by $C(u, v) = \dfrac{uv}{u + v - uv}$. [10]

(b) Show that this copula has lower tail dependence, but no upper tail dependence. [10]

4. A no claims discount system operates on a motor insurance portfolio with four levels of discount, 0% (state 1), 30% (state 2), 50% (state 3) and 60% (state 4). If a policyholder makes no claims during the year she moves up one discount level (or remains at the maximum discount level). If she makes one claim during the year, she moves down one discount level (or remains at the no discount level). If she makes two or more claims during the year, she moves directly, or remains at, the no discount level.

The probability that a policyholder makes no claims in one year is 0.9, while the probability that she makes one claim in one year is 0.09. Assume that policyholders enter the system on the 30% discount level.

(a) Write the matrix of transition probabilities. [05]

(b) What is the probability that a policyholder just entering the company will ever come back to the state of 30% discount? Justify your answer. [05]

(c) What is the expected discount during the third year a policyholder is in the company, that is, after the second contract renewal. [10]

(d) Explain why there exists a limiting distribution and obtain it. [10]

(e) Assume now that there are two classes of drivers, A and B. Policyholders classified as A enter the system in the 0% discount level and those classified as B enter the system in the 30% discount level. Knowing that the proportion of class A drivers is 20%, what is, for a randomly selected policyholder, the expected discount during the third year in the company? And in the long run (justify your answer)? [10]

5. Drivers arriving at a mechanical workshop wait on average 30 minutes while a first assessment on the vehicle is carried out (state A). After the first assessment, they are classified in two possible ways: simple repair, made just-in-time (state J) or need for further assessment (state F). Only 25% of the vehicles have just-in-time repair, the remaining needing further assessment. Vehicles needing only just-in-time repair take on average 1 hour to be discharged (state D), while vehicles needing further assessment take on average 45 minutes to be classified as light repair (state L), heavy repair but no need for new pieces (state N), and heavy repair needing new pieces (state P). From the vehicles needing further assessment, 5 in 10 go to state L, and 1 in 10 go to state N. Vehicles in state L take on average 4 hours to be discharged, those in state N take on average 24 hours to be discharged and those in state P take on average 48 hours to be discharged.

The workshop models the progress of the vehicles in the workshop by means of a time-homogeneous Markov process with states A, J, F, L, N, P and D. You can consider the time unit to be hours.

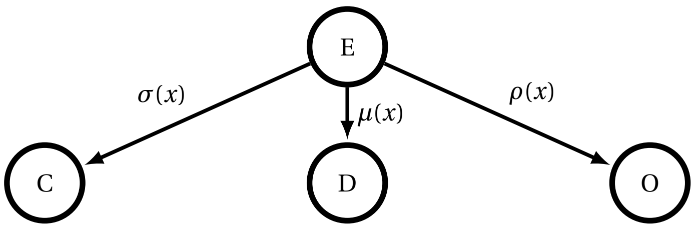
(a) Write down the infinitesimal generator matrix of the process. [10]

(b) What is the probability that a driver will wait more than 45 minutes to receive the information that the car needs either just-in-time repair or further assessment? [05]

(c) Using the forward differential equations derive an explicit expression for the probability that a newly arrived vehicle at time $t = 0$ is undergoing further assessment at time t. [10]

(d) Compute the expected time until closure of a vehicle that has just arrived to the workstation. [10]

6. The human resources department of an insurance company is concerned with the factors that drive its employees to leave the institution work force. They have considered the following non-homogeneous continuous time Markov model, $\{X(t)\}_{t\geqslant 0}$, where the states are: E - Employee; C - Left for a competitor company; D - Left for a different industry; O - Left for other reasons. In this model, they do not take into account the possibility of the employee returning to the company. Time is measured in years, corresponding to the age of individuals.



Denote $p_{ij}(x,t) = P(X(t) = j|X(x) = i)$, with $t > x$ and $i, j \in S = \{E, C, D, O\}$, and consider the following transition rates:

$\sigma(x) = 0.001 + 0.01e^{0.05x}$, $\mu(x) = 0.0005 + 0.01e^{-0.1x}$, and $\rho(x) = 0.0005 + 0.01e^{0.01x}$.

(a) Write the infinitesimal generator matrix of the process. [05]

(b) What is the probability that an employee aged 25 will remain in the company for the next 10 years? [10]

(c) Obtain an integral expression for the probability that an employee aged 55 will leave for a competitor company before she is 60. Explain all elements in your expression. Do not solve. [10]