Master in Actuarial Sciences

Probability and Stochastic Processes

07/01/2020

Time allowed: Three hours

1. (a) $P(X > 42\,000) = 1 - \Phi\left(\frac{\ln 42\,000 - 8.2}{2.1}\right) \approx 1 - \Phi(1.16) = 1 - 0.87698 = 0.12302$ [05]

$P(X > 40\,000) = 1 - \Phi\left(\frac{\ln 40\,000 - 8.2}{2.1}\right) \approx 1 - \Phi(1.14) = 1 - 0.87286 = 0.12714$

(b) $P(X > 42\,000 | X > 40\,000) = \dfrac{P(X > 42\,000, X > 40\,000)}{P(X > 40\,000)} = \dfrac{P(X > 42\,000)}{P(X > 40\,000)} = \dfrac{0.12302}{0.12714} = 0.9675948.$ [10]

(c) Let N be the number of large claims in a sample of 7 of such claims. [10]

Then $N \sim Binomial(n = 7, p = P(X > 40\,000) = 0.12714)$ and the desired probability is

$$P(N = 3) = \frac{7!}{3!4!} 0.12714^3 (1 - 0.12714)^4 = 0.04175351$$

(d) Let $X \sim LogNorm(\mu = 8.2, \sigma = 2.1)$ be the loss of a claim and M_{10} be the maximum loss of a sample of 10 such losses. Then [05]

$P(M_{10} > 42\,000) = 1 - P(M_{10} \leqslant 42\,000) = 1 - [F_X(42\,000)]^{10} = 1 - \left[\Phi\left(\dfrac{\ln 42\,000 - 8.2}{2.1}\right)\right]^{10} = 1 - 0.87698^{10} = 0.7309104.$

The probability that the maximum loss of 10 o such claims exceeds 42 000 is 73,1%.

2. (a) We have that [10]

- $f_1(x) = \frac{x}{9} e^{-x/3}$, $E[X_1] = \alpha\theta = 6$, $E[X_1^2] = \alpha(\alpha + 1)\theta^2 = 54$ and $Var[X_1] = \alpha\theta^2 = 18$
- $f_2(x) = \frac{1}{3} e^{-x/3}$, $F_2(x) = 1 - e^{-x/3}$, $S_2(x) = e^{-x/3}$, $E[X_2] = \theta = 3$, $E[X_2^2] = 2\theta^2 = 18$, $Var[X_2] = \theta^2 = 9$

and $f_Y(x) = \frac{3}{4} f_1(x) + \frac{1}{4} f_2(x)$ and $Z = \frac{3}{4} X_1 + \frac{1}{4} X_2$

$$E[Y] = \int_0^\infty x f_Y(x) dx = \int_0^{+\infty} x \left(\frac{3}{4} f_1(x) + \frac{1}{4} f_2(x)\right) dx = \frac{3}{4} \int_0^\infty x f_1(x) dx + \frac{1}{4} \int_0^\infty x f_2(x) dx = \frac{3}{4} E[X_1] + \frac{1}{4} E[X_2] = \frac{21}{4} = 5.25$$

$$E[Z] = E\left[\frac{3}{4} X_1 + \frac{1}{4} X_2\right] = \frac{3}{4} E[X_1] + \frac{1}{4} E[X_2] = \frac{21}{4} = 5.25$$

Thus $E[Y] = E[Z]$.

(b) [10]

$$E[Y^2] = \int_0^\infty x^2 f_Y(x) dx = \int_0^{+\infty} x^2 \left(\frac{3}{4} f_1(x) + \frac{1}{4} f_2(x)\right) dx = \frac{3}{4} \int_0^\infty x^2 f_1(x) dx + \frac{1}{4} \int_0^\infty x^2 f_2(x) dx = \frac{3}{4} E[X_1^2] + \frac{1}{4} E[X_2^2] = \frac{180}{4} = 45$$

$$Var[Y] = E[Y^2] - E^2[Y] = \frac{279}{16} = 17.4375$$

$$Var[Z] = Var\left[\frac{3}{4}X_1 + \frac{1}{4}X_2\right] = \frac{9}{16}Var[X_1] + \frac{1}{16}Var[X_2] = \frac{171}{16} = 10.6875$$

Thus $Var[Y] > Var[Z]$.

(c) We have that $S_1(x) = e^{-x/3} + \frac{x}{3}e^{-x/3}$, and $S_Y(y) = \frac{3}{4}S_1(x) + \frac{1}{4}S_2(x) = \frac{1}{4}e^{-x/3}(4+x)$. [10]

We have that $f_Y = \frac{3}{4}f_1(x) + \frac{1}{4}f_2(x) = \frac{1}{12}e^{-x/3}(x+1)$, hence $h_Y(x) = \dfrac{f_Y(x)}{S_Y(x)} = \dfrac{1}{3}\dfrac{1+x}{4+x}$.

$h'_Y(x) = \frac{1}{(4+x)^2} > 0$, thus the force of hazard is an increasing function of x meaning that Y is light tailed, according to this criteria.

(d) $q_{0.995} : F_2(q_{0.995}) = 0.995 \Leftrightarrow 1 - e^{-q_{0.995}/3} = 0.995 \Leftrightarrow q_{0.995} = 15.89495$. The 99.5% percentile of X_2 is $q_{0.995} = 15.89495$. [10]

$P(Y \leqslant 15.89495) = F_Y(15.89495) = \dfrac{3}{4}\left(1 - e^{-15.89495/3}\left(1 + \dfrac{15.89495}{3}\right)\right) + \dfrac{1}{4}\left(1 - e^{-15.89495/3}\right) = 0.9751313.$

The 99.5% percentile of X_2 corresponds approximately to the 97.5% percentile of Y, meaning that Y has a higher probability that events larger than 15.9 occur than X_2. When modeling losses, Y corresponds to a model with higher risk for large occurrences than X_2.

(e) [10]

$$f(x) = \begin{cases} p\,\dfrac{f_1(x)}{F_1(5)}, & 0 < x < 5 \\[2ex] (1-p)\,\dfrac{f_2(x)}{S_2(5)}, & x > 5 \end{cases}$$

In order to guarantee that this density is continuous, we need to guarantee the equality of both branches at $x = 5$:

$$p\,\frac{f_1(5)}{F_1(5)} = (1-p)\,\frac{f_2(5)}{S_2(5)} \Leftrightarrow p = 0.6119058$$

Thus

$$f(x) = \begin{cases} 0.6119058\dfrac{\frac{x}{9}e^{-x/3}}{1 - e^{-5/3}\left(1 + \frac{5}{3}\right)}, & 0 < x < 5 \\[3ex] 0.3880942\dfrac{\frac{1}{3}e^{-x/3}}{e^{-5/3}}, & x > 5 \end{cases}$$

3. (a) $F_X(x) = \lim_{y\to\infty} H(x,y) = (1+e^{-x})^{-1}$ and $F_Y(y) = \lim_{x\to\infty} H(x,y) = (1+e^{-y})^{-1}$. C is the copula of X and Y iff $H(x,y) = C(F_X(x), F_Y(y))$. [10]

$$C(F_X(x), F_Y(y)) = \frac{\frac{1}{(1+e^{-x})(1+e^{-y})}}{\frac{1}{(1+e^{-x})} + \frac{1}{(1+e^{-y})} - \frac{1}{(1+e^{-x})(1+e^{-y})}} = \frac{1}{1+e^{-x}+e^{-y}} = H(x,y)$$

(b) $\lambda_L = \lim\limits_{u\to 0}\dfrac{C(u,u)}{u} = \lim\limits_{u\to 0}\dfrac{\frac{u^2}{2u-u^2}}{u} = \lim\limits_{u\to 0}\dfrac{1}{2-u} = \dfrac{1}{2} \neq 0$, thus there is lower tail dependence. [10]

$\lambda_U = \lim\limits_{u\to 1}\dfrac{1-2u+C(u,u)}{1-u} = \lim\limits_{u\to 1}\dfrac{1-2u+\frac{u^2}{2u-u^2}}{1-u} = \lim\limits_{u\to 1}\dfrac{2-4u+2u^2}{2-2u+u^2} = 0$, thus there is no upper tail dependence.

4. (a) $P = \begin{pmatrix} 0.1 & 0.9 & 0 & 0 \\ 0.1 & 0 & 0.9 & 0 \\ 0.01 & 0.09 & 0 & 0.9 \\ 0.01 & 0 & 0.09 & 0.9 \end{pmatrix}$ [05]

(b) The chain is irreducible and finite, thus all states are positive recurrent and the probability of ever returning to any state is 1. Hence, the probability that a policyholder just entering the system will ever return to the 30% discount state is 1. [05]

(c) The expected discount after the second renewal is [10]

$$0P_{21}^2 + 0.3P_{22}^2 + 0.5P_{23}^2 + 0.6P_{24}^2 = 0 \times 0.019 + 0.3 \times 0.171 + 0.5 \times 0 + 0.6 \times 0.81 = 0.5373$$

(d) The chain is finite, irreducible and aperiodic (all states communicate and have the same period, and $d(1) = 1$). Hence, the chain is regular and has a unique limiting distribution given by the stationary distribution: [10]

$$\pi P = \pi \Leftrightarrow \begin{cases} 0.1\pi_1 + 0.1\pi_2 + 0.01\pi_3 + 0.01\pi_4 & = & \pi_1 \\ 0.9\pi_1 + 0.09\pi_3 & = & \pi_2 \\ 0.9\pi_2 + 0.09\pi_4 & = & \pi_3 \\ 0.9\pi_3 + 0.9\pi_4 & = & \pi_4 \end{cases}$$

with $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$, which leads to

$$\pi = \begin{bmatrix} 0.01300716 & 0.020405728 & 0.096658711 & 0.869928401 \end{bmatrix}$$

(e) The expected discount during the third year in the company for a randomly selected policyholder is: [10]

$$\begin{aligned} & 0.2(0P_{11}^2 + 0.3P_{12}^2 + 0.5P_{13}^2 + 0.6P_{14}^2) + 0.8(0P_{21}^2 + 0.3P_{22}^2 + 0.5P_{23}^2 + 0.6P_{24}^2) = \\ &= 0.2(0 \times 0.1 + 0.3 \times 0.09 + 0.5 \times 0.81 + 0.6 \times 0) + 0.8 \times 0.5373 = 0.2 \times 0.432 + 0.8 \times 0.573 = 0.51624 \end{aligned}$$

In the long run the probability that the policyholder will be in each state is given by the limiting distribution π, which is independent from the initial state. Thus, the expected discount for a randomly selected policyholder in the long run is

$$0 \times \pi_1 + 0.3 \times \pi_2 + 0.5 \times \pi_3 + 0.6 \times \pi_4 = 0.5764081$$

5. (a) [10]

$$\begin{array}{c} \\ A \\ J \\ F \\ Q = L \\ N \\ P \\ D \end{array} \begin{array}{c} \begin{array}{ccccccc} A & J & F & L & N & P & D \end{array} \\ \begin{pmatrix} -2 & \frac{1}{2} & \frac{3}{2} & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{4}{3} & \frac{2}{3} & \frac{2}{15} & \frac{8}{15} & 0 \\ 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & -\frac{1}{24} & 0 & \frac{1}{24} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{48} & \frac{1}{48} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

(b) $p_{AA}\left(\frac{3}{4}\right) = p_{\overline{AA}}\left(\frac{3}{4}\right) = e^{-2\frac{3}{4}} = 0.22313$ [05]

(c) From the forward differential equations we have that $p'_{AF}(t) = \frac{3}{2}p_{AA}(t) - \frac{4}{3}p_{AF}(t)$. We also have that $p_{AA}(t) = p_{\overline{AA}}(t) = e^{-2t}$, thus [10]

$$\begin{gathered} p'_{AF}(t) = \tfrac{3}{2}e^{-2t} - \tfrac{4}{3}p_{AF}(t) \Leftrightarrow p'_{AF}(t)e^{4t/3} + \tfrac{4}{3}p_{AF}(t)e^{4t/3} = \tfrac{3}{2}e^{-2t}e^{4t/3} \Leftrightarrow \left(p_{AF}(t)e^{4t/3}\right)' = \tfrac{3}{2}e^{-2t/3} \Leftrightarrow \\ \Leftrightarrow p_{AF}e^{4t/3} = -\tfrac{9}{3}e^{-2t/3} + C \Leftrightarrow p_{AF}(t) = -\tfrac{9}{4}e^{-2t} + Ce^{-4t/3} \end{gathered}$$

From the initial condition $p_{AF}(0) = 0$ we obtain $C = \frac{9}{4}$ and $p_{AF}(t) = \dfrac{9}{4}\left(e^{-4t/3} - e^{-2t}\right)$

(d) Let m_i be the expected time until reaching state D given that the chain is in state i. The required expected time is m_A. We have that $m_A = 0.5 + 0.25m_J + 0.75m_F$, $m_J = 1$, $m_F = 0.75 + 0.5m_L + 0.1m_N + 0.4m_P$, $m_L = 4$, $m_N = 24$, and $m_P = 48$. Thus $m_F = 24.35$ and $m_A = 19.0125$. [10]

6. (a) [05]

$$
Q = \begin{array}{c} \\ E \\ C \\ D \\ O \end{array}
\begin{array}{c}
\begin{array}{cccc} \quad E \quad & C & D & O \end{array} \\
\begin{pmatrix} -(\sigma(x)+\mu(x)+\rho(x)) & \sigma(x) & \mu(x) & \rho(x) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{array}
$$

$$
= \begin{array}{c} \\ E \\ C \\ D \\ O \end{array}
\begin{array}{c}
\begin{array}{cccc} \quad E \quad & C & D & O \end{array} \\
\begin{pmatrix} -0.002-0.01(e^{0.05x}+e^{-0.1x}+e^{0.01x}) & 0.001+0.01e^{0.05x} & 0.0005+0.01e^{-0.1x} & 0.0005+0.01e^{0.01x} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{array}
$$

(b) [10]

$$
\begin{aligned}
p_{\overline{EE}}(25,35) &= e^{-\int_{25}^{35}(0.002+0.01(e^{0.05x}+e^{-0.1x}+e^{0.01x}))dx} \\
&= e^{-0.002\times 10-0.2\left(e^{0.05\times 35}-e^{0.05\times 25}\right)-0.1\left(e^{-0.1\times 25}-e^{-0.1\times 35}\right)-\left(e^{0.01\times 35}-e^{0.01\times 25}\right)} \\
&= e^{-0.6130828} = 0.5416784
\end{aligned}
$$

(c) $p_{EC}(55,60) = \int_0^5 p_{EE}(55,55+s)\sigma(55+s)ds$ [10]

where

$$
p_{EE}(50,55+s) = e^{-0.002\times s-0.2\left(e^{0.05\times(55+s)}-e^{0.05\times 55}\right)-0.1\left(e^{-0.1\times 55}-e^{-0.1\times(55+s)}\right)-\left(e^{0.01\times(55+s)}-e^{0.01\times 55}\right)}
$$

and $\sigma(55+s) = 0.001+0.01e^{0.05(55+s)}$