LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

Master in Actuarial Sciences

Probability and Stochastic Processes

03/02/2020

Time allowed: Three hours

Instructions:

1. This paper contains 5 questions and comprises 3 pages including the title page.
2. Enter all requested details on the cover sheet.
3. You have 10 minutes of reading time. You must not start writing your answers until instructed to do so.
4. Number the pages of the paper where you are going to write your answers.
5. Attempt all 5 questions.
6. Begin your answer to each of the 5 questions on a new page.
7. Marks are shown in brackets. Total marks: 200.
8. Show calculations where appropriate.
9. An approved calculator may be used.
10. The distributed Formulary and the Formulae and Tables for Actuarial Examinations (the 2002 edition) may be used. **Note that, unless otherwise stated, the parametrisation used for the different distributions is that of the distributed Formulary.**

1. The loss per claim in a certain line of business is assumed to be normally distributed with mean $\mu = 750$ euros and standard deviation $\sigma = 235.607$ euros. Assume that claim amounts are independent.

(a) What is the probability that the maximum of 60 of such claims exceeds twice the expected value of one claim? [05]

(b) What is the probability that the aggregate loss of 100 of such claims exceeds 80 000 euros? [05]

(c) Suppose now that another actuary models each loss through an Inverse Weibull with parameters $\theta = 644.2028$ and $\tau = 5$, meaning that the expected loss is 750 and its standard deviation is 235.607. What is now the answer to the previous two questions? Compare the results. [10]

2. Consider the random variable X, representing losses in thousands of euros in a certain line of business, with probability density function given by:

$$f_X(x) = e^{-3x} + e^{-3x/2}, \quad x \geqslant 0.$$

(a) Show that X is a discrete mixture of two exponentials, identifying them and the mixture weights. [05]

(b) Indicate the moment generating function of X and use it to determine $E(X)$ and $V(X)$. [10]

(c) Determine the survival function of the equilibrium distribution of X. [10]

(d) Compute the mean excess loss of X and classify the tail of X according to that criteria. [10]

(e) How do you classify the tail of X based on moments? [05]

(f) Compute $E[(X-x)_+]$ and $E[X \wedge x]$ and verify their relation with the expectation of X, $E[X]$. [10]

(g) Determine the distribution of $Y = \theta X^{-1}$ and classify the parameter θ of the new distribution. [10]

3. Let X and Y be random variables with a joint distribution function given by

$$H_\alpha(x, y) = 1 - e^{-x} - e^{-y} + e^{-(x+y+\alpha xy)}, \qquad x \geqslant 0, y \geqslant 0, \qquad \alpha \in [0,1].$$

(a) Show that the marginal distribution functions are exponentials with mean 1 and that the corresponding copula is [10]

$$C_\alpha(u, v) = u + v - 1 + (1-u)(1-v)e^{-\alpha \ln(1-u)\ln(1-v)}$$

(b) Compute the conditional probability $P(X \leqslant x | Y = 1)$. [10]

4. The human resources department of a consulting firm is analysing the career progression of their employees in the company. Promotions are disclosed every year and there are five possible levels at the company: 1-Analyst; 2-Senior Analyst; 3-Consultant; 4-Associate Partner; 5-Partner. Assume that new employees always start as Analysts. The transition probability matrix is given as follows:

$$P = \begin{bmatrix} 0.55 & 0.4 & 0.05 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0.125 & 0.5 & 0.25 & 0.125 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

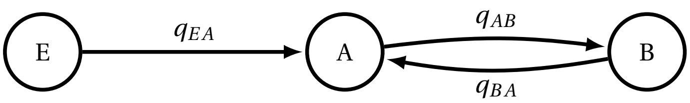
(a) Identify the communicating classes, indicating if they are open or closed, and classify the states according to period and recurrence. [05]

(b) What is the probability that an individual that is now a Consultant will be an Associate Partner in two years? And in three years? [10]

(c) Explain in words what does f_{22} represents and compute it. [10]

(d) What is the probability that an employee just entering the company becomes an Associate Partner? And a Partner? [10]

5. The career development strategy for certain employees of a financial institution states that all employees start as cash officers in an agency (state E), where they spend an average of 2 months. After this initial experience, they move to the department of risk analysis (state A). From here, it is the institution's policy that employees should also have experience in the department of business support (state B). After the mandatory initial experience as a cash officer, each employee remains an average of 1 year in the risk analysis department and 4 months in the business support one. The institution models the flow of employees through these three departments by means of an homogeneous Markov chain, as follows:



(a) Write the infinitesimal generator matrix of the process (use months as time unit). [05]

(b) What is the probability that an employee that is currently at the business support will remain there the next year (12 months). [05]

(c) Determine the proportion of time, in the long run, that an employee remains in each state. [10]

(d) Obtain the transition probability matrix in one step of the jump process, P^* (embedded Markov chain). [05]

(e) Compute $P^{*(2)}$ and $P^{*(3)}$. Is there a limiting distribution for the embedded Markov chain? [05]

(f) Obtain the stationary distribution of the embedded chain. Is it related to the stationary distribution of the continuous-time Markov process? [10]

(g) Interpret the meaning of $p_{EA}(t)$. Using the forward differential equations, show that [15]
$p'_{EA}(t) = \frac{1}{4} + \frac{1}{4} p_{EE}(t) - \frac{1}{3} p_{EA}(t)$ and find an explicit expression for $p_{EA}(t)$ by solving this equation.

(h) Provide an integral expression for the probability that an individual that is just entering the company will be in the area of business support in one year. Detail all elements in your expression, but do not solve it. [10]