LISBON SCHOOL OF ECONOMICS & MANAGEMENT
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Master in Actuarial Sciences

Probability and Stochastic Processes

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1. (a) Let M_{60} be the maximum loss of 60 of such claims. Then [10]

$$P(M_{60} > 1500) = 1 - [F_X(1500)]^{60} = 1 - \left[\Phi\left(\frac{1500-750}{235.607}\right)\right]^{60} = 1 - 0.9992719^{60} = 1 - 0.9572383 = 0.04276167$$

(b) We have that $X_i \sim N(\mu = 750, \sigma = 318.5)$, hence $\sum_{i=1}^{100} X_i \sim N\left(100\mu, \sqrt{100\sigma^2}\right)$. Thus [05]

$$P\left(\sum_{i=1}^{100} X_i > 80\,000\right) = 1 - \Phi\left(\frac{80\,000 - 75\,000}{2356.07}\right) = 1 - \Phi(2.12) = 1 - 0.982997 = 0.01700302$$

(c) In this case, [10]

$$P(M_{60} > 1500) = 1 - [F_X(1500)]^{60} = 1 - \left[e^{-(644.2028/1500)^5}\right]^{60} = 1 - 0.985496^{60} = 1 - 0.4161901 = 0.5838099$$

Now, the probability that the maximum of 60 losses exceeds twice the expected loss is approximately 58%, more than 10 times the same probability using the normal model with same mean and variance. Both models have the same expectation and standard deviation, but the Inverse Weibull has higher probability for the occurrence of extreme values.

Using central limit theorem (CLT), we have that $\sum_{i=1}^{100} X_i \sim^a N\left(100E[X] = 100 \times 750, \sqrt{100Var[X]} = 10 \times 235.607\right)$, thus the probability that the aggregate loss of 100 of such claims is higher than 80000 is the same as before:

$$P\left(\sum_{i=1}^{100} X_i > 80\,000\right) \underbrace{\approx}_{CLT} 1 - \Phi\left(\frac{80\,000 - 75000}{2356.07}\right) = 1 - \Phi(2.12) = 0.01700302$$

2. (a) $f_X(x) = \frac{1}{3}3e^{-3x} + \frac{2}{3}\frac{3}{2}e^{-\frac{3x}{2}}$, thus X is the mixture of $X_1 \sim Exp\left(\frac{1}{3}\right)$ with $X_2 \sim Exp\left(\frac{2}{3}\right)$, with mixture weights $\frac{1}{3}$ and $\frac{2}{3}$, respectively. [05]

(b) $M_X(t) = \frac{1}{3}M_1(t) + \frac{2}{3}M_2(t) = \frac{1}{3}\left(1 - \frac{t}{3}\right)^{-1} + \frac{2}{3}\left(1 - \frac{2t}{3}\right)^{-1}$, and [10]
$M_X'(t) = \frac{1}{9}\left(1 - \frac{t}{3}\right)^{-2} + \frac{4}{9}\left(1 - \frac{2t}{3}\right)^{-2}$, and $M_X''(t) = \frac{2}{27}\left(1 - \frac{t}{3}\right)^{-3} + \frac{16}{27}\left(1 - \frac{2t}{3}\right)^{-3}$.
$E[X] = M_X'(0) = \frac{5}{9}$, $E[X^2] = M_x''(0) = \frac{18}{27}$, and $Var[X] = E[X^2] - E^2[X] = \frac{29}{81}$.

(c) $S_X(x) = \frac{1}{3}S_1(x) + \frac{2}{3}S_2(x) = \frac{1}{3}e^{-3x} + \frac{2}{3}e^{-3x/2}$. [10]

$$\int_x^{\infty} S_X(t)dt = \frac{1}{9}e^{-3x} + \frac{4}{9}e^{-3x/2}, \text{ thus } S_e(x) = \frac{\int_x^{\infty} S_X(t)dt}{E[X]} = \frac{e^{-3x} + 4e^{-3x/2}}{5}.$$

(d) $e_X(x) = \dfrac{\int_x^\infty S_X(t)dt}{S_X(x)} = \dfrac{1}{3}\dfrac{e^{-3x}+4e^{-3x/2}}{e^{-3x}+2e^{-3x/2}}$, and $e_X'(x) = \dfrac{e^{-9x/2}}{\left(e^{-3x}+2e^{-3x/2}\right)^2} > 0$. [10]

The mean excess loss is an increasing function of x, thus X is heavy tailed according to this criteria.

(e) X has moment generating function, thus, based on moments, it is light tailed. [05]

(f) $E[(X-x)_+] = \int_x^\infty S_X(t)dt = \frac{1}{9}e^{-3x} + \frac{4}{9}e^{-3x/2}$ and $E[X\wedge x] = \int_0^x S_X(t)dt = \frac{1}{9}\left(1-e^{-3x}\right) + \frac{4}{9}\left(1-e^{-3x/2}\right)$. [10]

$E[(X-x)_+] + E[X\wedge x] = \frac{5}{9} = E[X]$.

(g) $P\left(Y \leqslant y\right) = P\left(X \geqslant \frac{\theta}{y}\right) = S_X\left(\frac{\theta}{y}\right) = \frac{1}{3}e^{-\frac{3\theta}{y}} + \frac{2}{3}e^{-\frac{3\theta}{2y}}$. [10]

Considering $Z = \frac{Y}{\theta}$, $P(Z \leqslant z) = P(Y \leqslant z\theta) = \frac{1}{3}e^{-\frac{3}{z}} + \frac{2}{3}e^{-\frac{3}{2z}}$. The distribution of $Z = \frac{Y}{\theta}$ is independent of θ ($Z = X^{-1}$), hence θ is a scale parameter for Y.

3. (a) $F_X(x) = \lim_{y\to\infty} H_\alpha(x,y) = 1-e^{-x} \sim Exp(1)$ and $F_Y(y) = \lim_{x\to\infty} H_\alpha(x,y) = 1-e^{-y} \sim Exp(1)$. [10]

C_α is the copula of X and Y iff $H_\alpha(x,y) = C_\alpha(F_X(x), F_Y(y))$.

$$C_\alpha(F_X(x), F_Y(y)) = 1-e^{-x}+1-e^{-y}-1+e^{-x}e^{-y}e^{\alpha\ln e^{-x}\ln e^{-y}} = 1-e^{-x}-e^{-y}+e^{-x}e^{-y}e^{\alpha xy} = H_\alpha(x,y)$$

(b) $P(X\leqslant x|Y=1) = \frac{\partial}{\partial v}C_\alpha(u,v)\Big|_{(u,v)=(F_X(x),F_Y(1))}$, thus $P(X\leqslant x|Y=1) = 1-e^{-x(1+\alpha)}(1+\alpha x)$, since [10]

$\frac{\partial}{\partial v}C_\alpha(u,v) = 1+(1-u)e^{-\alpha\ln(1-u)\ln(1-v)}(\alpha\ln(1-u)-1)$.

4. (a) The communicating classes are {1}, {2,3}, {4} and {5}. {1}, {2,3} are open, thus states 1, 2 and 3 are transient, while {4} and {5} are closed, hence states 4 and 5 are (positive, since the chain is finite) recurrent. States 4 and 5 are actually absorbing states. Regarding the period, $d(1) = d(3) = d(4) = d(5) = 1$ and $d(2) = 1$ since it communicates with 3. [05]

(b) $P_{34}^2 = 0.375$ and $P_{34}^3 = 0.46875$ [10]

(c) f_{22} is the probability of ever returning to state 2. We have that $f_{22} = \sum_{i=1}^{n} f_{22}^{(n)}$, where $f_{22}^{(n)}$ is the probability that the first return to state 2 occurs in exactly n steps. [10]

In this case $f_{22}^{(1)} = 0$ and $f_{22}^{(n)} = \left(\frac{1}{2}\right)^{n-2}\frac{1}{8}$, for $n\geqslant 2$. Thus $f_{22} = \sum_{i=2}^{n}\left(\frac{1}{2}\right)^{n-2}\frac{1}{8} = \frac{1}{8}\sum_{i=0}^{n}\left(\frac{1}{2}\right)^{n} = \frac{1}{8}\frac{1}{1-\frac{1}{2}} = \frac{1}{4}$.

(d) Let u_i be the probability that the chain is absorbed in 4 given it is in state i. The probability that an employee just entering the company becomes an Associate Partner is u_1. We have $u_4 = 1$ and $u_5 = 0$ and using first step analysis: [10]

$$\begin{cases} u_1 &= 0.55u_1 + 0.4u_2 + 0.05u_3 \\ u_2 &= u_3 \\ u_3 &= 0.125u_2 + 0.5u_3 + 0.25 \end{cases}$$

Thus $u_1 = u_2 = u_3 = \frac{2}{3}$, that is, the probability that an employee becomes and Associate Partner is $2/3$. The probability that an employee becomes a Partner is then $1-2/3 = 1/3$.

5. (a) [05]

$$\begin{array}{cc} & \begin{array}{ccc} E & A & B \end{array} \\ \begin{array}{c} E \\ Q=A \\ B \end{array} & \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{12} & \frac{1}{12} \\ 0 & \frac{1}{4} & -\frac{1}{4} \end{pmatrix} \end{array}$$

(b) $p_{\overline{BB}}(12) = e^{-12/4} = 0.049787068$ [05]

(c) The limiting distribution, π, of the continuous process is given by [10]

$$\pi Q = 0 \Leftrightarrow \left[\begin{array}{ccc} \pi_E & \pi_A & \pi_B \end{array}\right] \left[\begin{array}{ccc} -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{12} & \frac{1}{12} \\ 0 & \frac{1}{4} & -\frac{1}{4} \end{array}\right] = \left[\begin{array}{ccc} 0 & 0 & 0 \end{array}\right] \Leftrightarrow \left\{\begin{array}{rcl} -\frac{1}{2}\pi_E & = & 0 \\ \frac{1}{2}\pi_E - \frac{1}{12}\pi_A + \frac{1}{4}\pi_B & = & 0 \\ \frac{1}{12}\pi_A - \frac{1}{4}\pi_B & = & 0 \end{array}\right.$$

which results in $\pi_E = 0$ and $\pi_A = 3\pi_B$. From $\pi_E + \pi_A + \pi_B = 1$ we obtain $\pi = \left[\begin{array}{ccc} 0 & \frac{3}{4} & \frac{1}{4} \end{array}\right]$, corresponding to the probability that an employee is in each state in the long run. Notice that $\pi_E = 0$, since E is a transient state.

(d) [05]

$$P^* = \begin{array}{c} \\ E \\ A \\ B \end{array} \begin{array}{c} \begin{array}{ccc} E & A & B \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

(e) $P^{*(2)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $P^{*(3)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = P^*$. We have that $P^{*(3)} = P^* \neq P^{*(2)}$, hence $P^{*(n)} = P^*$ [05] if n is odd, and $P^{*(n)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, if n is even. Thus, there is not a limiting distribution of the embedded Markov chain.

(f) The stationary distribution of P^* is π^* given by [10]

$$\pi^* P^* = \pi^* \Leftrightarrow \left[\begin{array}{ccc} \pi_E^* & \pi_A^* & \pi_B^* \end{array}\right] \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc} \pi_E^* & \pi_A^* & \pi_B^* \end{array}\right] \Leftrightarrow \left\{\begin{array}{rcl} -\frac{1}{2}\pi_E^* & = & 0 \\ \frac{1}{2}\pi_E^* - \frac{1}{12}\pi_A^* + \frac{1}{4}\pi_B^* & = & 0 \\ \frac{1}{12}\pi_A^* - \frac{1}{4}\pi_B^* & = & 0 \end{array}\right.$$

which results in $\pi_E^* = 0$ and $\pi_A^* = \pi_B^*$. From $\pi_E^* + \pi_A^* + \pi_B^* = 1$ we obtain $\pi^* = \left[\begin{array}{ccc} 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right]$.

The stationary distribution of the embedded process gives the proportion of time that the system jumps into state i, without taking into account the holding time in each state. Since the holding time in state i is on average $\frac{1}{-q_{ii}}$, we have that the proportion of time the continuous process spends in each state is $\pi_i = \left(\frac{\pi^*}{-q_{ii}}\right) \Big/ \left(\sum_j \frac{\pi_j^*}{-q_{jj}}\right)$. In this case

$$\pi_E = \pi_E^* = 0, \quad \pi_A = \frac{0.5 \times 12}{0.5 \times 12 + 0.5 \times 4} = \frac{3}{4}, \quad \pi_B = \frac{0.5 \times 4}{0.5 \times 12 + 0.5 \times 4} = \frac{1}{4}.$$

(g) $p_{EA}(t)$ is the probability that an individual that is a cash officer at instant $t = 0$ will be in the risk analysis department at instant t. From the forward differential equations we have [15]

$p'_{EA}(t) = \frac{1}{2}P_{EE}(t) - \frac{1}{12}p_{EA}(t) + \frac{1}{4}p_{EB}(t)$. Using the fact that $p_{EB}(t) = 1 - p_{EE}(t) - p_{EA}(t)$, we obtain

$$p'_{EA}(t) = \frac{1}{4} + \frac{1}{4}p_{EE}(t) - \frac{1}{3}p_{EA}(t).$$

We have that $p_{EE}(t) = p_{\overline{EE}}(t) = e^{-\frac{t}{2}}$, since after leaving state E it is not possible to return. Thus

$$p'_{EA}(t) = \frac{1}{4} + \frac{1}{4}e^{-t/2} - \frac{1}{3}p_{EA}(t) \Leftrightarrow p'_{EA}(t) + \frac{1}{3}p_{EA}(t) = \frac{1}{4} + \frac{1}{4}e^{-t/2}$$

multiplying by $e^{t/3}$, we obtain

$$\left(e^{t/3}p_{EA}(t)\right)' = \frac{1}{4}e^{t/3} + \frac{1}{4}e^{-t/6} \Leftrightarrow e^{t/3}p_{EA}(t) = \frac{3}{4}e^{t/3} - \frac{6}{4}e^{-t/6} + C \Leftrightarrow p_{EA}(t) = \frac{3}{4} - \frac{6}{4}e^{-t/2} + Ce^{-t/3}$$

From the initial conditions $p_{EA}(0) = 0$ which means $C = \frac{3}{4}$. Thus

$$p_{EA}(t) = \frac{3}{4} - \frac{3}{4}\left(2e^{-t/2} - e^{-t/3}\right)$$

(h) $p_{EB}(12) = \int_0^{12} p_{EA}(12-s)\, q_{AB}\, p_{\overline{BB}}(s)\, ds$ [10]

where

$p_{EA}(12-s) = \frac{3}{4} - \frac{3}{4}\left(3e^{-(12-s)/3} - 2e^{-(12-s)/2}\right)$,

$q_{AB} = \frac{1}{12}$ and

$p_{\overline{BB}}(s) = e^{-s/4}$.