## 2 Functions of several variables: Topology, limits and continuity

**2.1.** Determine the domain of the following functions and represent it graphically.

$a)f(x,y)=\dfrac{\sqrt{9-x^2-y^2}}{1-\ln x}$ $\quad b)f(x,y)=\dfrac{\sqrt{e-e^x}}{\ln(4-x^2-y^2)}$ $\quad c)f(x,y)=\ln(x-y)\sqrt{(y-x)(x^2+y^2-1)}$

$d)f(x,y)=\dfrac{\sqrt[3]{x+y}}{\ln x^2-\ln(3-x)^2}$ $\quad e)f(x,y)=\ln(x-y)^2$

**Solution:** a) $\{(x,y)\in\mathbb{R}^2 : x>0\wedge x\neq e\wedge x^2+y^2\leq 9\}$ b) $\{(x,y)\in\mathbb{R}^2 : x\leq 1\wedge x^2+y^2<4\wedge x^2+y^2\neq 3\}$ c) $\{(x,y)\in\mathbb{R}^2 : x-y>0\wedge x^2+y^2-1\leq 0\}$ d) $\{(x,y)\in\mathbb{R}^2 : x\neq 0\wedge x\neq 3\wedge x\neq 3/2\}$ e) $\{(x,y)\in\mathbb{R}^2 : x\neq y\}$

**2.2.** Determine the interior, exterior and boundary of the following subsets of $\mathbb{R}^2$. Classify them with respect to being open, closed and bounded.

$A=\{(x,y)\in\mathbb{R}^2 : (x-1)^2+(y+2)^2<4\}$ $\quad B=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\geq 9\}$

$C=\{(x,y)\in\mathbb{R}^2 : \dfrac{x^2}{9}+\dfrac{y^2}{16}\leq 1\}$ $\quad D=\{(x,y)\in\mathbb{R}^2 : \dfrac{(x-2)^2}{4}+\dfrac{(y+1)^2}{9}=1\}$

**Solution:** $\text{Int}(A)=A$, $\text{Bdy}(A)=\{(x,y)\in\mathbb{R}^2 : (x-1)^2+(y+2)^2=4\}$, $ext(A)=\{(x,y)\in\mathbb{R}^2 : (x-1)^2+(y+2)^2>4\}$, $A$ is open and bounded. $\text{Int}(B)=\{(x,y)\in\mathbb{R}^2 : x^2+y^2>9\}$, $\text{Bdy}(B)=\{(x,y)\in\mathbb{R}^2 : x^2+y^2=9\}$, $\text{Ext}(B)=\{(x,y)\in\mathbb{R}^2 : x^2+y^2<9\}$, $B$ is closed and not bounded. $\text{Int}(C)=\{(x,y)\in\mathbb{R}^2 : \frac{x^2}{9}+\frac{y^2}{16}<1\}$, $\text{Bdy}(C)=\{(x,y)\in\mathbb{R}^2 : \frac{x^2}{9}+\frac{y^2}{16}=1\}$, $\text{Ext}(C)=\{(x,y)\in\mathbb{R}^2 : \frac{x^2}{9}+\frac{y^2}{16}>1\}$, $C$ is closed and bounded. $\text{Int}(D)=\emptyset$, $fr(D)=D$, $\text{Ext}(D)=\{(x,y)\in\mathbb{R}^2 : \frac{(x-2)^2}{4}+\frac{(y+1)^2}{9}\neq 1\}$, $D$ is closed and bounded.

**2.3.** Represent graphically and analitically the the domain $D_f$, as well as $\text{Int}(D_f)$, $\text{Bdy}(D_f)$ and $D'_f$. State in each case if $D_f$ open, closed, bounded and compact.

$a)f(x,y)=\dfrac{|x|-4}{\ln(4-x^2-y^2)}$ $\quad b)f(x,y)=\sqrt{x(1-x)}+\dfrac{\ln(x^2-y)}{\sqrt{y+x}}$

$c)f(x,y)=\sqrt{y+x-1}.\ln(4-(x+1)^2-(y-1)^2)$ $\quad d)f(x,y)=\sqrt{x-|y|}+\sqrt{2-y^2-x}$

$e)f(x,y)=x\sqrt{y^2-4}+\sqrt[4]{16-x^2-y^2}$

**Solution:** a) $D_f = \{(x, y) \in \mathbb{R}^2 : 4-x^2-y^2 > 0 \wedge \ln(4-x^2-y^2) \neq 0\} = \{(x, y) \in \mathbb{R}^2 : x^2+y^2 < 4 \wedge x^2+y^2) \neq 3\}$, $\text{Int}(D_f) = D_f$, $\text{Bdy}(D_f) = \{(x, y) \in \mathbb{R}^2 : x^2+y^2 = 4 \vee x^2+y^2 = 3\}$, $D'_f = \{(x, y) \in \mathbb{R}^2 : x^2+y^2 \leq 4\}$. $D_f$ is open, not closed, bounded, not compact.

b) $D_f = \{(x, y) \in \mathbb{R}^2 : x(1-x) \geq 0 \wedge x^2-y > 0 \wedge y+x > 0\} = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \wedge -x < y < x^2\}$, $\text{Int}(D_f) = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1 \wedge -x < y < x^2\}$, $\text{Bdy}(D_f) = \{(x, y) \in \mathbb{R}^2 : (y = x^2 \wedge 0 \leq x \leq 1) \vee (y = -x \wedge 0 \leq x \leq 1) \vee (x = 1 \wedge -x \leq y \leq x^2)\}$, $D'_f = \{(x, y) \in \mathbb{R}^2 : (0 \leq x \leq 1) \wedge (-x \leq y \leq x^2)\}$. $D_f$ is not open nor closed, bounded, not conmpact.

c) $D_f = \{(x, y) \in \mathbb{R}^2 : y + x - 1 \geq 0 \wedge 4 - (x + 1)^2 - (y - 1)^2 > 0\} = \{(x, y) \in \mathbb{R}^2 : y \geq 1 - x \wedge (x + 1)^2 + (y - 1)^2 < 4\}$, $\text{Int}(D_f) = \{(x, y) \in \mathbb{R}^2 : y > 1 - x \wedge (x + 1)^2 + (y - 1)^2 < 4\}$, $\text{Bdy}(D_f) = \{(x, y) \in \mathbb{R}^2 : y = 1 - x \wedge (x + 1)^2 + (y - 1)^2 \leq 4\} \cup \{(x, y) \in \mathbb{R}^2 : y \geq 1 - x \wedge (x + 1)^2 + (y - 1)^2 = 4\}$, $D'_f = \{(x, y) \in \mathbb{R}^2 : y \geq 1-x \wedge (x+1)^2+(y-1)^2 \leq 4\}$. $D_f$ is not open nor closed, bounded, not compact.

d)$D_f = \{(x, y) \in \mathbb{R}^2 : x-|y| \geq 0 \wedge 2-y^2-x \geq 0\}$, $\text{Int}(D_f) = \{(x, y) \in \mathbb{R}^2 : x-|y| > 0 \wedge 2-y^2-x > 0\}$, $\text{Bdy}(D_f) = \{(x, y) \in \mathbb{R}^2 : x - |y| = 0 \wedge 2 - y^2 - x \geq 0\} \cup \{(x, y) \in \mathbb{R}^2 : x - |y| \geq 0 \wedge 2 - y^2 - x = 0\}$, $D'_f = D_f$. $D_f$ is not open, is closed, bounded, compact.

e)$D_f = \{(x, y) \in \mathbb{R}^2 : y^2 - 4 \geq 0 \wedge 16 - x^2 - y^2 \geq 0\} = \{(x, y) \in \mathbb{R}^2 : (y \geq 2 \vee y \leq -2) \wedge x^2 + y^2 \leq 16\}$, $\text{Int}(D_f) = \{(x, y) \in \mathbb{R}^2 : (y > 2 \vee y < -2) \wedge x^2 + y^2 < 16\}$, $\text{Bdy}(D_f) = \{(x, y) \in \mathbb{R}^2 : (y = 2 \vee y = -2) \wedge x^2 + y^2 \leq 16\} \cup \{(x, y) \in \mathbb{R}^2 : (y \geq 2 \vee y \leq -2) \wedge x^2 + y^2 = 16\}$, $D'_f = D_f$. $D_f$ is not open, is closed, bounded, compact.

**2.4.** Determine the domain $D_f$ of the function defined by

$$f(x, y) = \frac{\ln(x^2 + y^2 - 4)}{|x| - 4}.$$

Represent $D_f$ graphically and show that it is open and unbounded. Check if $\text{Ext}(D_f)$ is also open and unbounded.

**Solution:** $D_f = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 - 4 > 0 \wedge |x| - 4 \neq 0\}$. $Ext(D_f) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 4\}$.

**2.5.** Determine the interior, the boundary and the limit points of the following subsets of $\mathbb{R}^2$.

$a)A = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$

$b)B = \{(x, y) \in \mathbb{R}^2 : (x, y) = (\frac{1}{n}, \frac{1}{n}), n \in \mathbb{N}\}$

$c)C = \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ y = (-1)^n \frac{1}{n}, n \in \mathbb{N}\}.$

**Solution:** a) $\text{Int}(A) = \emptyset, \text{Bdy}(A) = A, A' = \emptyset$. b) $\text{Int}(B) = \emptyset$, $\text{Bdy}(B) = B \cup \{(0, 0)\}$, $B' = \{(0, 0)\}$. c) $\text{Int}(C) = \emptyset$, $\text{Bdy}(C) = C \cup \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ y = 0\}$, $C' = fr(C)$.

**2.6.** Compute the limits of the following sequences, or show that they no not exist.

$$\text{a) } \lim \left( \left(\frac{2n^2+3}{1+2n^2}\right)^{n^2}, \ln\left(\frac{2n}{2n+1}\right)^{n+\frac{1}{2}} \right) \quad \text{b) } \lim \left( n^3+n-n^2-1), \sqrt{n}.\frac{\sqrt{n}+3}{(\sqrt{n}+1)^2} \right)$$

$$\text{c) } \lim \left( n\left(e^{\frac{1}{n}}-1\right), \sin\frac{n\pi}{2} \right)$$

**Solution:** a) $(e, -\frac{1}{2})$; b) $(+\infty, 1)$; c) does not exist.

**2.7.** Compute the following limits:

$$\text{a) } \lim \overline{x}_n, \text{ com } \overline{x}_n = \left[ \frac{n}{2n+1}, \left(1+\frac{2}{n}\right)^n, \ln\left(1+\frac{1}{n}\right)^n \right]$$

$$\text{b) } \lim \overline{x}_n \text{ com } \overline{x}_n = \left[ \sqrt{n}-\sqrt{n-1}, \left(\sqrt[n]{e}-1\right).n, n.\ln\frac{n+2}{n}, \left(1-\frac{n^2}{n^2+1}\right)^{\frac{1}{3}} \cdot \left(\frac{n^2+1}{2n^2}\right)^{\frac{1}{3}} \right]$$

**Solution:** a) $(1/2, e^2, 1)$; b) $(0, 1, 2, 0)$.

**2.8.** For each of the following functions, investigate the existence of limit at the point $(0, 0)$.

a) $f(x,y) = \dfrac{x^2-y^2}{x(x+y)}$ b) $f(x,y) = \dfrac{x^3-y^3}{x^2+y^2}$ c) $f(x,y) = \dfrac{x^2+y}{\sqrt{x^2+y^2}}$

d) $f(x,y) = xy\dfrac{x^2-y^2}{x^2+y^2}$ e) $f(x,y) = \dfrac{xy}{x^2+y^2}$ f) $f(x,y) = \dfrac{x^2y}{x^4+y^2}$

g) $f(x,y) = \dfrac{x^2(x+y)}{x^2+y^2}$ h) $f(x,y) = \dfrac{x^2-y^2+2x^3}{x^2+y^2}$, i) $f(x,y) = \begin{cases} \dfrac{y}{x}\sqrt{x^2+y^2}, & \text{se } x>0,\ y>0 \\ 0, & \text{se } x<0 \text{ ou } y<0. \end{cases}$

**Solution:** a) does not exist b) 0 c)does not exist d) 0 e) does not exist f) does not exist g) 0 h) does not exist i) does not exist.

**2.9.** Determine, if possible, a continuous extention of each of the functions in the previous exercise to the point $(0, 0)$.

**Solution:** When the limit does not exist it is not possible to define such extention. In the other cases simply define $f(0,0) = \lim_{(x,y)\to(0,0)} f(x,y)$.

**2.10.** Investigate the existence of the following limits

$$a) \lim_{(x,y)\to(2,1)} \frac{(x-2)(y-1)}{(x-2)^2+(y-1)^2} \qquad b) \lim_{(x,y)\to(1,-3)} \frac{\sqrt{(x-1)(y+3)}+\sin(x-1)(y+3)}{\sqrt{(x-1)(y+3)}}$$

$$c) \lim_{(x,y)\to(2,1)} \frac{3(x-2)^2(y-1)}{(x-2)^2+(y-1)^2} \qquad d) \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x}$$

$$e) \lim_{(x,y)\to(1,0)} \frac{x^2-y^2-1}{x-1} \qquad f) \lim_{(x,y)\to(0,0)} \frac{x^2y+x^2+y^2}{x^2+y^2}$$

$$g) \lim_{(x,y)\to(0,1)} \frac{x^2\sqrt{|y-1|}}{x^2+(y-1)^2} \qquad h) \lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2}$$

**Solution:** a) does not exist  b) 1  c) 0  d) does not exist  e) does not exist  f) 1  g) 0  h) 0.

**2.11.** Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = \dfrac{x^3-y^3}{x-y}$.

a) Determine the domain $D_f$..

b) Show that $x^3 - y^3 = (x-y)p(x,y)$ , where $p(x,y)$ is a polynomial.

c) Can $f$ be extended by continuity to the line $y = x$?

**Solution:** a) $D_f = \mathbb{R}^2 \setminus \{(x,x) : x \in \mathbb{R}\}$. c) yes, just define $f(x,x) = 3x^2$.

**2.12.** Consider the function $f(x,y) = \frac{x^2y}{x^2-y^2}$.

a) Compute $\lim_{\substack{(x,y)\to(0,0),\\ y=x+x^2}} f(x,y)$. What can you conclude about $\lim_{(x,y)\to(0,0)} f(x,y)$ ?

b) Compute $\lim_{\substack{(x,y)\to(0,0),\\ y=mx}} f(x,y)$, $|m| \neq 1$. What can you conclude about $\lim_{(x,y)\to(0,0)} f(x,y)$ ?

**Solution:**

a) $-\frac{1}{2}$. If it exists $\lim_{(x,y)\to(0,0)} f(x,y)$, is $-\frac{1}{2}$.  b) 0. It does not exist $(0 \neq -\frac{1}{2})$.