Matematics II

2014/2015

Economics and Management

Exercises with solutions

1 Complements of Linear Algebra

1.1. Determine the eigenvalues of each of the following matrices and, if they are real, determine the corresponding eigenvectors together with the algebraic and geometric multiplicities.

$$\text{a)} \begin{bmatrix} 2 & -7 \\ 3 & -8 \end{bmatrix} \qquad \text{b)} \begin{bmatrix} 2 & 4 \\ -2 & 6 \end{bmatrix} \qquad \text{c)} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\text{d)} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{e)} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \qquad \text{f)} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 3 & 0 \\ -2 & 5 & 1 \end{bmatrix}$$

$$\text{g)} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad \text{h)} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{i)} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Solution: a) $\lambda_1 = -5,\ a.m. = 1$; eigenvectors $u = (c, c)$, with $c \neq 0,\ g.m. = 1$;
$\lambda_2 = -1,\ a.m. = 1$; eigenvectors $u = (7/3c, c)$, with $c \neq 0,\ g.m. = 1$;
b) The characteristic polynomial does not have real eigenvalues ($\lambda_1 = 4 + 2i,\ \lambda_2 = 4 - 2i$);
c) $\lambda_1 = 1 + \sqrt{2};\ a.m. = 1$; eigenvectors $u = ((1 + \sqrt{2})c, c)$, with $c \neq 0,\ g.m. = 1$;
$\lambda_2 = 1 - \sqrt{2};\ a.m. = 1$; eigenvectors $u = ((1 - \sqrt{2})c, c)$, with $c \neq 0,\ g.m. = 1$;
d) $\lambda = 0,\ a.m. = 2$; eigenvectors $u = (c, 0)$, with $c \neq 0,\ g.m. = 1$;
e) $\lambda_1 = 2,\ a.m. = 1$; eigenvectors $u = (c, 0, 0)$, with $c \neq 0,\ g.m. = 1$;
$\lambda_2 = 3,\ a.m. = 1$; eigenvectors $u = (0, c, 0)$, with $c \neq 0,\ g.m. = 1$;
$\lambda_3 = 4,\ a.m. = 1$; eigenvectors $u = (0, 0, c)$, with $c \neq 0,\ g.m. = 1$;
f) $\lambda_1 = 3,\ a.m. = 2$; eigenvectors $u = (-c, 0, c)$, with $c \neq 0,\ g.m. = 1$;
$\lambda_2 = -1,\ a.m. = 1$; eigenvectors $u = (c, 0, c)$, with $c \neq 0,\ g.m. = 1$;
g) $\lambda_1 = 0,\ a.m. = 2$; eigenvectors $u = (-c_1 - c_2, c_1, c_2)$, with $c_1^2 + c_2^2 \neq 0$, $g.m. = 2$;
$\lambda_2 = 3,\ a.m. = 1$; eigenvectors $u = (c, c, c)$, with $c \neq 0,\ g.m. = 1$;
h) $\lambda = 1,\ a.m. = 3$; eigenvectors $u = (c, 0, 0)$, with $c \neq 0,\ g.m. = 1$;
i) $\lambda_1 = 2,\ a.m. = 2$; eigenvectors $u = (c_1, c_1, c_2)$, with $c_1^2 + c_2^2 \neq 0,\ g.m. = 2$;
$\lambda_2 = 0,\ a.m. = 1$; eigenvectors $u = (-c, c, 0)$, with $c \neq 0,\ g.m. = 1$.

1.2. Show that

a) Every eigenvalue of A is also an eigenvalue of A^T.

b) If λ is an eigenvalue of A and $|A| \neq 0$ then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

c) If λ is an eigenvalue of A then λ^k is an eigenvalue of A^k, $k \in \mathbb{N}$.

1.3. Let $A = \begin{bmatrix} 1 & 4 \\ 6 & -1 \end{bmatrix}$

a) Determine the eigenvalues and eigenvectors of A.

b) Determine the eigenvalues and eigenvectors of A^{100}.

Solution: a) $\lambda_1 = -5$, eigenvectors $u = (-2/3c, c)$, with $c \neq 0$, ;
$\lambda_2 = 5$, eigenvectors $u = (c, c)$, with $c \neq 0$, ;
b) $\lambda = 5^{100}$, eigenvectors $u = (-2/3c_1, c_1) + (c_2, c_2)$, with c_1, c_2 not simultaneously zero;

1.4. Consider a matrix A and a vector $\boldsymbol{x}$ given by

$$A = \begin{bmatrix} a & a & 0 \\ a & a & 0 \\ 0 & 0 & b \end{bmatrix} \qquad \text{e} \qquad \boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad (a, b \in \mathbb{R})$$

a) Write down the characteristic polynomial $p(\lambda)$.

b) Determine the eigenvalues and eigenvectors of matrix A.

c) Compute $\boldsymbol{x}^T A \boldsymbol{x}$.

d) Consider that $a, b \geq 0$. Without performing any calculations show that there exists a a vector $\boldsymbol{x} \neq \boldsymbol{0}$ such that $\boldsymbol{x}^T A \boldsymbol{x} = 0$

e) Classify the quadratic form $\boldsymbol{x}^T A \boldsymbol{x}$ for all possible values of a, b.

Solution: a) $P(\lambda) = (b - \lambda)(-\lambda)(2a - \lambda)$; b) $\lambda_1 = b, \lambda_2 = 0, \lambda_3 = 2a$; c) $\mathbf{x}^T A\mathbf{x} = ax^2 + 2axy + ay^2 + bz^2$;
e) PSD if $(a \geq 0 \ e \ b \geq 0)$; NSD if $(a \leq 0 \ e \ b \leq 0)$;
Ind. in the remaining cases, i.e. $(a > 0 \ e \ b < 0)$ ou $(a < 0 \ e \ b > 0)$.

1.5. Classify the following quadratic Forms:

a) $q(x,y) = x^2 + 2xy + y^2$

b) $q(x,y) = x^2 - 2xy + y^2$

c) $q(x,y) = x^2 - y^2$

d) $q(x,y,z) = x^2 + 4xy - 2xz + 7y^2 - 3z^2$

e) $q(x,y,z) = x^2 - 4xy + 4xz - z^2$

f) $q(x,y) = 6x^2 + 4xy + 3y^2$

g) $q(x,y) = x^2 + 4xy + y^2$

h) $q(x,y) = 2x^2 + 6xy + 4y^2$

i) $q(x,y,z) = 3y^2 + 4xz$

j) $q(x,y) = x^2 + 4xy + ay^2, \quad a \in \mathbb{R}$

Solution: a)PSD b)PSD c)Ind. d)Ind. e)Ind. f)PD g)Ind. h)Ind. i)Ind. j)Ind. if $a < 4$, PSD if $a = 4$ and PD if $a > 4$.

1.6. Classify the following symmetric matrices (with respect to beeing PD, PSD, ND, NSD, Ind.)

a) $\begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$ b) $\begin{bmatrix} -5 & 1 \\ 1 & 5 \end{bmatrix}$ c) $\begin{bmatrix} -5 & 1 \\ 1 & -5 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 2 \\ 2 & a \end{bmatrix}, a \in \mathbb{R}$ e) $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 7 & 0 \\ -1 & 0 & -3 \end{bmatrix}$

f) $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 7 & -5 \\ -1 & -5 & 4 \end{bmatrix}$ g) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$ h) $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & a & 0 \\ 0 & a & 2 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}, a \in \mathbb{R}$ i) $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -3 & 0 \\ 1 & 0 & -2 \end{bmatrix}$

j) $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Solution: a) PD b) Ind. c) ND d)Ind. if $a < 4$, PSD if $a = 4$ and ND if $a > 4$ e) Ind. f) PSD g) Ind. h)Ind. if $a < -\sqrt{2} \vee a > \sqrt{2}$, PD if $-\sqrt{2} < a < \sqrt{2}$ and PSD if $a = \pm\sqrt{2}$ i) ND j) Ind.