3 Differential Calculus in $\mathbb{R}^n$

3.1.

Determine the first order partial derivatives of the following functions, defining them in the largest possible domain.

a) $f(x,y,z) = 3xy + x^2 - zy + z^2$; b) $f(x,y) = \begin{cases} x^2 - yx, & y \neq x \\ x, & y = x. \end{cases}$

Solution: a) $f'_x = 3y + 2x$; $f'_y = 3x - z$; $f'_z = -y + 2z$; $\mathcal{D}f'_x = \mathcal{D}f'_y = \mathcal{D}f'_z = \mathbb{R}^3$
b) $f'_x = 2x - y$ if $x \neq y$ and $f'_x = 0$ if $x = y = 0$; $f'_y = -x$ if $x \neq y$ and $f'_y = 0$ if $x = y = 0$
$\mathcal{D}f'_x = \mathcal{D}f'_y = \mathbb{R}^2 \backslash \{(a,a) : a \neq 0\}$

3.2. Show that $f(x,y) = \dfrac{x-y+1}{x+y}$ is a solution of the equation

$$\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} = \frac{2}{x+y}$$

in any of the sets defined by $x + y > 0$ or $x + y < 0$.

3.3. Consider the function

$$f(x,y) = \begin{cases} \dfrac{e^{x-y} - (x-y+1)}{x-y}, & x \neq y \\ 0, & x = y \end{cases}$$

a) Discuss the continuity of $f(x,y)$ at $(1,1)$.

b) Check that $f'_x(a,a) + f'_y(a,a) = 0, \ \forall a \in \mathbb{R}$.

Solution:

a) f is continuous at $(1,1)$.

3.4. Given the function

$$f(x,y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & , x^2 + y^2 \neq 0 \\ 0 & , x = y = 0 \end{cases},$$

compute the directional derivatives at $(0,0)$, whenever they exist.

Solution: $\partial_{\vec{v}} f(0,0)$ exists for $\vec{v} = (\alpha, \alpha)$ and $\vec{v} = (\alpha, -\alpha)$, $\alpha \in \mathbb{R}\setminus\{0\}$. In that case, $\partial_{\vec{v}} f(0,0) = 0$.

3.5. Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & , x \neq 0 \\ 0 & , x = 0 \end{cases},$$

a) Show that f admits a directional derivative at $(0,0)$ along any direction and compute it.

b) Show that f is not continuous at $(0,0)$.

c) Without performing any calculations, state the value of $\dfrac{\partial f}{\partial x}(0,0)$ and of $\dfrac{\partial f}{\partial y}(0,0)$.

Solution: a) $\partial_{(\alpha,\beta)} f(0,0) = \begin{cases} \dfrac{\beta^2}{\alpha} & \text{se } \alpha \neq 0 \\ 0 & \text{se } \alpha = 0 \end{cases}$, $(\alpha,\beta) \in \mathbb{R}^2\setminus\{(0,0)\}$. c) $\dfrac{\partial f}{\partial x}(0,0) = \dfrac{\partial f}{\partial y}(0,0) = 0$.

3.6.

Study the differentiability of the following functions at the proposed points and obtain the expression of the first order differentials (in case they are differentiable).

a) $f(x,y) = x^2 + y^2$, at point (0,0);

b) $f(x,y) = \begin{cases} x+y, & x \neq y \\ x+1, & x = y \end{cases}$, at (1,1);

c) $f(x,y) = \begin{cases} xy - 2y + 3x, & x \neq y \\ x^2y^2 + 3x - 2y, & x = y \end{cases}$, at (0,0);

d) $y = (x^2+1, x)$, at $x = 1$;

Solution:
a) f is differentiable at $(0,0)$; $Df(0,0)(\mathbf{h}) = 0$;
b) f is not differentiable at em $(1,1)$;
c) f is differentiable at $(0,0)$; $Df(0,0)(\mathbf{h}) = 3h_1 - 2h_2$;
d) y is differentiable at $x = 1$; $Df(1)(\mathbf{h}) = (2h, h)$.

3.7. Write down the expressions of the first order differentials of each given function, at the proposed points:

a) $f(x,y) = y^x$, at a generic point (a,b), with $b > 0$;

b) $f(x_1,x_2,x_3) = \dfrac{x_1 - x_2 + x_3}{\sqrt{x_3 - 1}}$, at (1,-3,2).

Note: Admit that the functions are differentiable.

Solution: a) $Df(a,b)(\mathbf{h}) = b^a \log b.h_1 + ab^{a-1}.h_2$ b) $Df(1,-3,2)(\mathbf{h}) = h_1 - h_2 - 2h_3$.

3.8. Show that the following functions are continuous but not differentiable at the given points:

a) $f(x,y) = \begin{cases} \dfrac{-3x(y-2)^2 + x^3}{x^2 + (y-2)^2}, & \text{if } (x,y) \neq (0,2) \\ 0, & \text{if } (x,y) = (0,2), \end{cases}$, at $(0,2)$.

b) $g(x,y) = \begin{cases} \dfrac{2xy}{\sqrt{x^2+y^2}}, & \text{if } x^2+y^2 \neq 0 \\ 0, & \text{if } x = y = 0, \end{cases}$ at $(0,0)$

c) $h(x,y) = \sqrt{|x|}\cos y$, at $(0,0)$.

3.9. Use the chain rule to compute

a) $\dfrac{df}{dt}$, where $f = x^2y^3$, knowing that $x = te^t$ e $y = t^2 + 1$;

b) $\dfrac{df}{dt}$, where $f = u^2 + v^3$, knowing that $u = \dfrac{x}{y}$, $v = (x+2y)^3$ e $x = \dfrac{1}{t}$, $y = tg\ t$;

c) $\dfrac{dz}{dt}$, knowing that $z = \dfrac{2xy}{x^2+y^2}$ e $x = \cos t$, $y = \sin t$.

d) $\nabla f(1,1)$, where $f(x,y) = \sin(2u - v^3 + w)$, knowing that $u = e^{x^2-y}$, $v = xy^2$ e $w = x^3y^2$;

e) $\dfrac{\partial f}{\partial y}(0,1,1)$, where $f(x,y,z) = (u^2 - 3v)^5$, knowing that $u = e^{\frac{xy}{z}}$ e $v = \ln(y^2z^3)$;

f) $\nabla f(1,2,3)$, where $f(x,y,z) = g(u,v,w)$, with $u = 5x + 3z$, $v = 8x + 2y$, $w = -y + z$ and knowing that $\nabla g(14,12,1) = (4,5,6)$.

Solution:
a) $\frac{df}{dt} = 2te^{2t}(t+1)(t^2+1)^3 + 6t^3e^{2t}(t^2+1)^2$;
b) $\frac{df}{dt} = -2\frac{1}{t^3}\frac{1}{tg^2\ t} - 2\frac{1}{t^2}\frac{\sec^2 t}{tg^3\ t} + 9(-\frac{1}{t^2} + 2\sec^2 t)(\frac{1}{t} + 2tg\ t)^8$; c) $\frac{dz}{dt} = 2 - 4\sin^2 t$; d) $\nabla f(1,1) = (4\cos 2, -6\cos 2)$; e) $\frac{\partial f}{\partial y}(0,1,1) = -30$; f) $\nabla f(1,2,3) = (60,4,18)$.

3.10. If a function $f(u,v,w)$ is differentiable at $u = x - y$, $v = y - z$ and $w = z - x$, show that setting $F(x,y,z) = f(x-y, y-z, z-x)$ we have

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} = 0.$$

3.11. Consider the function

$$g(x,y)=\begin{cases}\dfrac{(x-1)^2y^2}{(x-1)^2+y^2}, & (x,y)\neq(1,0)\\ 0, & (x,y)=(1,0)\end{cases}$$

a) Determine the partial derivatives $g'_x(x,y)$ and $g'_y(x,y)$, as well as their domain of definition.

b) Show that $g'_x(x,y)$ and $g'_y(x,y)$ are continuous over $\mathbb{R}^2$.

c) Study the differentiability of f at $(1,0)$.

d) Discuss the continuity of f at $(1,0)$.

Solution:

a) $g'_x(x,y)=\begin{cases}\dfrac{2(x-1)y^4}{((x-1)^2+y^2)^2}, & (x,y)\neq(1,0)\\ 0, & (x,y)=(1,0)\end{cases}$ $\quad g'_y(x,y)=\begin{cases}\dfrac{2(x-1)^4y}{((x-1)^2+y^2)^2}, & (x,y)\neq(1,0)\\ 0, & (x,y)=(1,0)\end{cases}$

Therefore, $D_{g'_x}=D_{g'_y}=\mathbb{R}^2$. c) g is differentiable at $(1,0)$. d) g continuous at (1,0).

3.12. Let $f:\mathbb{R}^2\to\mathbb{R}$ be a function defined by

$$f(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{\sqrt{x^2+y^2}}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0).\end{cases}$$

a) Compute $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$.

b) Determine $\dfrac{\partial f}{\partial y}(x,y)$ and show that it is discontinuous at $(0,0)$.

c) Check that$f$ is differentiable at $(0,0)$.

d) Compute $\partial_{(\frac{3}{5},\frac{4}{5})}f(0,0)$.

e)Discuss the continuity of f at $(0,0)$.

Solution:

a) $f'_x(0,0)=f'_y(0,0)=0$;

b)$f'_y(x,y)=\begin{cases}2y\sin\frac{1}{\sqrt{x^2+y^2}}-y\frac{1}{\sqrt{x^2+y^2}}\cos\frac{1}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$; d) 0; e) f is continuous at $(0,0)$.

3.13. Use the function

$$g(x,y) = \begin{cases} \dfrac{\sin x}{y}, & y \neq 0 \\ 0, & y = 0 \end{cases}$$

and the point $(0,0)$ to show that a function with finite partial derivatives at a given point is not necessarily continuous at that point. Is the given function differentiable at $(0,0)$? Why?

Solution:

The function is not continuous at $(0,0)$, and so it is also not differentiable.

3.14. Considerer the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{|x|+|y|}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

a) Show that f is continuous at $(0,0)$.

b) Determine $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$.

c) Show that f is not differentiable at (0,0). Without performing any calculations, what can you conclude about the continuity of $\dfrac{\partial f}{\partial x}$ e $\dfrac{\partial f}{\partial y}$ at (0,0)?

Solution:
b) $f'_x(0,0) = f'_y(0,0) = 0$ c) Since f is not differentiable at $(0,0)$, at least one of the functions f'_x or f'_y is not continuous at $(0,0)$.

3.15. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} \dfrac{x(x-y)}{x+y} & \text{if } x+y \neq 0 \\ 0 & \text{if } x+y = 0 \end{cases}$$

a) Study the continuity of f at $(0,0)$

b) Compute the partial derivative $\dfrac{\partial f}{\partial x}$ and discuss its continuity at $(0,0)$.

c) Study the differentiability of f at $(0,0)$.

d) Show that $\left(\dfrac{\partial f}{\partial x}(0,0), \dfrac{\partial f}{\partial y}(0,0)\right) \cdot \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \neq \delta_{\vec{(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})}} f(0,0)$. Comment on the result.

Solution:

a) f is not continuous at (0,0).
b) $f'_x(x,y) = \frac{x^2+2xy-y^2}{(x+y)^2}$ if $x+y \neq 0$ and $f'_x(x,y) = 1$ if $(x,y) = (0,0)$ (it does not exist $f'_x(a,-a), a \neq 0$); $f'_x(x,y)$ is not continuous at $(0,0)$.
c) f is not differentiable at $(0,0)$.
d) The two values onlym had to be equal if f was differentiable at $(0,0)$.

3.16. Compute $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^4 f}{\partial x^2 \partial z \partial y}$, for $f(x,y,z) = z^2x^2y + xye^z$.

Solution: $\dfrac{\partial^2 f}{\partial x^2} = 2yz^2$, $\dfrac{\partial^4 f}{\partial x^2 \partial z \partial y} = 4z$.

3.17. Compute f''_{x^2}, f''_{xy} and f'''_{xyx} for each of the following functions, indicating the corresponding domain of definition:

$$a)\ f(x,y) = x\sin(x+y); \qquad b)\ f(x,y) = \begin{cases} y\sin x, & y \neq 0 \\ 2, & y = 0 \end{cases}$$

Solution:

a) $f''_{x^2} = 2\cos(x+y) - x\sin(x+y)$, $f''_{xy} = \cos(x+y) - x\sin(x+y)$ and $f'''_{xyx} = -2\sin(x+y) - x\cos(x+y)$.
b) $f''_{x^2} = -y\sin x$, $f''_{xy} = \cos x$ and $f'''_{xyx} = -\sin x$;

3.18. Compute the differential of order 2, 3 and 4 of the function $f(x,y) = \sqrt{xy}$ at $(1,1)$.

Solution:
$D^2 f(1,1)(\mathbf{h}^2) = -\frac{1}{4}h_1^2 + \frac{1}{2}h_1h_2 - \frac{1}{4}h_2^2$,
$D^3 f(1,1)(\mathbf{h}^3) = \frac{3}{8}h_1^3 - \frac{3}{8}h_1^2h_2 - \frac{3}{8}h_1h_2^2 + \frac{3}{8}h_2^3$,
$D^4 f(1,1)(\mathbf{h}^4) = -\frac{15}{16}h_1^4 + 4\frac{3}{16}h_1^3h_2 + 6\frac{1}{16}h_1^2h_2^2 + 4\frac{3}{16}h_1h_2^3 - \frac{15}{16}h_2^4$.

3.19. Determine the differential of order n of the function $f(x,y) = \sin(x+y)$, at the point (0,0).

Solution: $D^n f(0,0)(\mathbf{h}) = \sin(n\pi/2) \displaystyle\sum_{i=0}^{n} \binom{n}{i} h_1^i h_2^{n-i}$

3.20. Show that $f(x,y) = log(e^x + e^y)$ satisfies the (differential) equation

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 0$$

everywhere in $\mathbb{R}^2$.

3.21. Let $f \in C^2(\mathbb{R}^2)$ be a real function such that $\dfrac{\partial f}{\partial u}(0,0) = \dfrac{\partial f}{\partial v}(0,0) = 1$. Also, let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by

$$g(x, y) = f(y\text{sen } x, y^2).$$

Show that the Hessian matrix of g at $(0,0)$ is given by $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$.

3.22. Consider $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x, y) = xy^2 + g(u, v, w), \text{ com } u = \text{sen } y^2, \ v = \ln x \text{ e } w = ye^x.$$

Assumingt that g is of class $C^2(\mathbb{R}^3)$, compute $\dfrac{\partial^2 f}{\partial y \partial x}(1,0)$.

Solution: $\frac{\partial^2 f}{\partial y \partial x}(1,0) = e\left(\frac{\partial^2 g}{\partial w \partial v}(0,0,0) + \frac{\partial g}{\partial w}(0,0,0)\right)$.

3.23. Show that the following functions are homogeneous or positively homogeneous. Determine in each case the degree of homogeneity and verify Euler's identity.

a) $f(x, y) = \log \dfrac{(x+y)^2}{xy}$ b) $f(x, y, z) = \dfrac{\sqrt{x^2+y^2}}{z^2}$ c) $f(x, y) = \begin{cases} (x+y)\sin\left(\dfrac{xy}{x^2+y^2}\right), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$.

Solution:

a) f is homogeneous with degree 0;
b) f is positively homogeneous with degree -1;
c) f is homogeneous with degree 1.

3.24. Study the function $g(x, y, z) = x^2 + x^\alpha y^{\beta-3} - z^{3\alpha} y^\beta$ e de $h(x, y) = \dfrac{x^3 y^\alpha + x^{\beta-1}}{y^{3-\beta}}$, with respect to its homogeneity in terms of the parameters $\alpha, \beta \in \mathbb{R}$,

a) Using the definition.

b) Using Euler's identity.

Solution:

g is homogeneous with degree 2 for $\alpha = -\frac{3}{2}$ and $\beta = \frac{13}{2}$;
h is homogeneous with degree $\alpha + \beta$ for $\beta = \alpha + 4, \alpha \in \mathbb{R}$.

3.25. Assuming that $g(u,v)$ is differentiable $\left(\dfrac{x}{y}, \dfrac{z}{x}\right)$, with $x, y \neq 0$, show that

$$f(x,y,z) = x^2.g\left(\frac{x}{y}, \frac{z}{x}\right),$$

satisfies the identity $x\, f'_x + y\, f'_y + z\, f'_z = 2.f$. Interpret this results in terms of homogeneity.

Solution: f is positively homogeneous with degree 2.

3.26. Let $f(\mathbf{x}) : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ be an homogeneous, nonconstant function with degree 0. Show that $\lim\limits_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x})$ does not exist.

3.27. Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ definida por

$$f(x,y) = \ln\left(\frac{xy}{x+y}\right).$$

Write down Taylor's formula with degree 2, around (1,1).

Solution: $\ln\dfrac{(1+h)(1+k)}{2+h+k} \;=\; -\ln 2 + \dfrac{1}{2}h + \dfrac{1}{2}k + \dfrac{1}{2}\left(-\dfrac{3}{4}h^2 + \dfrac{1}{2}hk - \dfrac{3}{4}k^2\right) + r_3(h,k).$