4 Optimization Problems

4.1. Determine and classify the critical points of the following functions from $\mathbb{R}^2$ to $\mathbb{R}$.

$a)x^2+y^2$	$b)x^2-y^2$	$c)x^3+y^3$	$d)x^3-y^3$
$e)x^4+y^4$	$f)x^4-y^4$	$g)3xy-x^3-y^3$	$h)x\ln x+y\ln y$
$i)x^3+ye^y$	$j)2x^3+xy^2+5x^2+y^2$	$k)x^4+y^4-4xy+1$	$l)x^2y^2$

Solution: a) $(0,0)$ is a minimum point; b) c) d) $(0,0)$ is a saddle point ; e) $(0,0)$is a minimum point; f) $(0,0)$ is a saddle point; g) $(0,0)$ is a saddle point and $(1,1)$ é maximizante; h) $(1/e,1/e)$ is a minimum point; i) $(0,-1)$ is a saddle point; j) $(0,0)$ is a minimum point, $(-5/3,0)$ é maximizante, $(-1,2)$ e $(-1,-2)$ are saddle points; k) $(0,0)$ is a saddle point, $(1,1)$ e $(-1,-1)$ are minimum points; l) $(0,b)$ e $(a,0)$ $\forall a,b\in\mathbb{R}$, are minimum points;

4.2. Determine and classify the critical points of the following functions, in terms of the parameter $a\in\mathbb{R}\setminus\{0\}$

$a)f(x,y)=e^{x^2-ay^2}$	$b)f(x,y)=ax^2-y^2$
$c)f(x,y)=x^3-ax^2-3y^2$	$d)f(x,y)=\frac{16}{5}x^5+ay^2-x$

Solution: a) Critical point: $(0,0)$. if $a<0$, minimum point; if $a>0$, saddle point. b) Critical point: $(0,0)$. If $a>0$, $(0,0)$ is a saddle point; if $a<0$, $(0,0)$ is a maximum point. c) Critical points: $(0,0)$ and $(\frac{2a}{3},0)$. If $a>0$, $(0,0)$ is a maximum point and $(\frac{2a}{3},0)$ is a saddle point; if $a<0$, $(0,0)$ is a saddle point and $(\frac{2a}{3},0)$ is a maximum point. d) Critical points: $(-\frac{1}{2},0)$ and $(\frac{1}{2},0)$. if $a<0$, $(-\frac{1}{2},0)$ is a maximum point and $(\frac{1}{2},0)$ is a saddle point; if $a>0$, $(-\frac{1}{2},0)$ is a saddle point and $(\frac{1}{2},0)$ is a minimum point.

4.3. Consider the function $f(x,y)=(y-\alpha)\,xe^x$.

a) Knowing that $(0,1)$ is a critical point, determine α and classify this critical point.

b) Show that f is unbounded.

Solution: a) $\alpha=1$. The critical point is a saddle point.

4.4. Let function $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ be defined by $f(x,y)=4\alpha(y-2)^2+(\beta^2-1)(2x-2)^2$, where $\alpha\neq 0$, $\beta\neq 1$, $\beta\neq -1$. Show thate $(1,2)$ is the only critical point and classify it in terms of all possible values of α and β.

Solution: If $|\beta| < 1$ and $\alpha < 0$ then $(1,2)$ is a local maximum; if $|\beta| > 1$ and $\alpha > 0$ then $(1,2)$ is a local minimum; in all other cases it is a saddle point.

4.5. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $f(x,y) = x^2 e^{y^3-3y}$.

a) Determine all critical points of function f.

b) Show that f attains its global minimum at points of the form $(0,b)$.

c) Justify that

(i) f is unbounded over $\mathbb{R}^2$;

(ii) f has a maximum and minimum over $B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 9\}$.

Solution: a) Critical points: $(0,b)$ with $b \in \mathbb{R}$.

4.6. Determine the global extrema of f over the set M, where

$$a) f(x,y,z) = x - 2y + 2z, \qquad M = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

$$b) f(x,y) = 4x^2 + y^2, \qquad M = \{(x,y) \in \mathbb{R}^2 : 2x^2 + y^2 = 1\}$$

$$c) f(x,y) = xy, \qquad M = \left\{(x,y) \in \mathbb{R}^2 : \tfrac{x^2}{8} + \tfrac{y^2}{2} = 1\right\}$$

$$d) f(x,y,z) = x^2 + 2y - 2z, \qquad M = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 8\}$$

$$e) f(x,y) = x^2 + 2xy + y^2, \qquad M = \{(x,y) \in \mathbb{R}^2 : (x-3)^2 + y^2 = 2\}$$

$$f) f(x,y,z) = 2x + 2y^2 + z^2, \qquad M = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 2\}$$

$$g) f(x,y,z) = e^{-x^2-y^2}, \qquad M = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

$$h) f(x,y) = 4xy - 2x^2 - 2y^2, \qquad M = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

$$i) f(x,y) = x^2 + 2xy + y^2, \qquad M = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 8\}$$

Solution: a) max. = 3, min.=-3; b) max.= 2, min. = 1; c) max. = 2, min. = -2; d) max. =10, min. = -8; e) max. = 25, min. = 1; f) max. = 9/2, min. = $-2\sqrt{2}$. g) max. = 1, min. = $1/e$; h) max. = 0, min. = -4; i) max. = 16, min.= 0

4.7. Determine the global extrema of f over the set A, where

$a) f(x,y,z) = x - 2y + 2z,\quad A = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 1\}$
(note: compare with a) from previous exercise)

$b) f(x,y) = 4x^2 + y^2,\quad A = \{(x,y) \in \mathbb{R}^2 : 2x^2+y^2 \le 1\}$
(note: compare with b) from previous exercise)

$c) f(x,y) = x^2 + 2xy + y^2,\quad A = \{(x,y) \in \mathbb{R}^2 : (x-3)^2+y^2 \le 2\}$
(note: compare with e) from previous exercise)

Solution: a) max. $= 3$, min. $= -3$; b) max. $= 2$, min. $=$ é 0; c) max. $= 25$, min. $= 1$.

4.8. Determine the maximum and minimum distance to the origin of the points in the ellipse $5x^2 + 6xy + 5y^2 = 8$.

Solution: The maximum distance is 2 and the minimum distance is 1.

4.9. Solve the optimization problem $\min\,(x + 4y + 3z)$ subject to the condition $x^2 + 2y^2 + \frac{1}{3}z^2 = b$, $(b > 0)$.

Solution: The minimum value is $-6\sqrt{b}$, attained at $(-\frac{\sqrt{b}}{6}, -\frac{\sqrt{b}}{3}, -\frac{3\sqrt{b}}{2})$.

4.10. Determine the point in the ellipse $x^2 + 2xy + 2y^2 = 2$ with smallest x coordinate.

Solution: $(-2, 1)$.

4.11. determine teh global extrema of $f(x,y) = e^{x^2+y^2+z^2}$ over the set $\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1, z = 2 - y\}$.

Solution: The maximum value is e^{10}, attained at $(0,-1,3)$; the minimum is e^2, attained at $(0,1,1)$.