6 Differential equations

6.1. Determine the general solution of the following differential equations with separable variables.

$a) y' = xy - x$ $\qquad$ $b)\ dx\, e^y = dy(x+1) - dx$ $\qquad$ $c) \frac{dy}{dx} + \frac{1+y^3}{xy^2(1+x^2)} = 0$

$d) \sqrt{1-x^2}dy - \sqrt{1-y^2}dx = 0$ $\qquad$ $e) e^{x^4} yy' = x^3\left(9+y^4\right)$ $\qquad$ $f) e^y(4+x^2)y' = x\left(2+e^y\right)$

$g) e^{3x}dy + \left(4+y^2\right)dx = 0$ $\qquad$ $h) 4xe^y dx + \left(x^4+4\right)dy = 0$

Solution: a) $y(x) = 1 + e^{\frac{1}{2}x^2}C$ b) $\ln\left(e^{y(x)}+1\right) - y(x) + \ln(x+1) = C$ c) $\frac{1}{3}\ln|1+y^3| + \ln|x| - \frac{1}{2}\ln(1+x^2) = C$ d) $\arcsin(y(x)) - \arcsin x = C$ e) $\frac{1}{6}\arctan\left(\frac{1}{3}y^2(x)\right) + \frac{1}{4}e^{-x^4} = C$ f) $\ln\left(2+e^{y(x)}\right) - \frac{1}{2}\ln\left(4+x^2\right) = C$ g) $\frac{1}{2}\arctan\left(\frac{1}{2}y(x)\right) - \frac{1}{3}e^{-3x} = C$ h) $-e^{-y(x)} + \arctan\frac{1}{2}x^2 = C$

6.2. Solve the following initial value problems.

a) $y' + 4y = 0, \quad y(0) = 6$ $\qquad$ b) $\dfrac{dy}{dt} + y\sin t = 0, \quad y(\pi/3) = 3/2$

c) $(1+x^2)y' + y = 0, \quad y(1) = 1$ $\qquad$ d) $2y' + 4xy = 4x, \quad y(0) = -2$

e) $y' + y\sin x = \sin x\cos x, \quad y(\frac{\pi}{2}) = 0$

Solution: a) $y(x) = 6e^{-4x}$ b) $y(t) = \frac{3}{2}e^{\cos t - \frac{1}{2}}$ c) $y(x) = e^{\frac{\pi}{4} - \arctan(x)}$ d) $y(x) = 1 - 3e^{-x^2}$ e) $y(x) = \cos x + 1 - e^{\cos x}$.

6.3. show that any homogeneous first order linear differential equation can be written as a

6.4. Determine de general solution of the following differential equations.

$a) y'' - 7y' + 12y = 0$ $\qquad$ $b) y'' + 4y = 0$ $\qquad$ $c) y'' - 4y' + 4y = 0$

$d) y'' + 2y' + 10y = 0$ $\qquad$ $e) y'' + y' - 6y = 8$ $\qquad$ $f) y'' + 3y' + 2y = e^{5x}$

$g) y'' - y = \sin x$ $\qquad$ $h) y'' - y = e^{-x}$ $\qquad$ $i) y'' - 6y = 36(x-1)$

$j) y'' - 9y = 9x^2$ $\qquad$ $k) y'' + 3y' + 2y = \sin x$ $\qquad$ $l) y'' + 3y' + 2y = e^{-x}$

$m) y'' - 4y' + 4y = 6e^{2x}$

Solution: a) $y(x) = C_1e^{3x} + C_2e^{4x}$ b) $y(x) = C_1\cos 2x + C_2\sin 2x$ c) $y(x) = (C_1 + C_2x)e^{2x}$ d) $y(x) = (C_1\cos 3x + C_2\sin 3x)e^{-x}$ e) $y(x) = C_1e^{-3x} + C_2e^{2x} - \frac{4}{3}$ f) $y(x) = C_1e^{-2x} + C_2e^{-x} + \frac{1}{42}e^{5x}$ g) $y(x) = C_1e^{-x} + C_2e^{x} - \frac{1}{2}\sin x$ h) $y(x) = C_1e^{x} + C_2e^{-x} - \frac{1}{2}xe^{-x}$ i) $y(x) = C_1e^{\sqrt{6}x} + C_2e^{-\sqrt{6}x} - 6x + 6$ j) $y(x) = C_1e^{3x} + C_2e^{-3x} - x^2 - \frac{2}{9}$ k) $y(x) = C_1e^{-2x} + C_2e^{-x} - \frac{3}{10}\cos x + \frac{1}{10}\sin x$ l) $y(x) = C_1e^{-2x} + C_2e^{-x} + xe^{-x}$ m) $y(x) = C_1e^{2x} + C_2xe^{2x} + 3x^2e^{2x}$.

6.5. Solve the following initial value problems.

$a)\begin{cases} y'' + y' - 2y = 0 \\ y(0) = -1, y'(0) = 1 \end{cases}$ $b)\begin{cases} y'' + 2y' + 5y = 0 \\ y(0) = 0, y'(0) = 1 \end{cases}$ $c)\begin{cases} y'' + 2y' + y = x^2 \\ y(0) = 0,\ y'(0) = 1 \end{cases}$

$d)\begin{cases} y'' + 4y = 4x + 1 \\ y(\frac{\pi}{2}) = 0,\ y'(\frac{\pi}{2}) = 0 \end{cases}$ $e)\begin{cases} 9y'' + y = 0 \\ y\left(\frac{3}{2}\pi\right) = 2,\ y'\left(\frac{3}{2}\pi\right) = 0 \end{cases}$ $f)\begin{cases} 2y'' - 4y' + 2y = 0 \\ y(0) = -1, y'(0) = 1 \end{cases}$

$g)\begin{cases} y'' - 2y' + 10y = 10x^2 \\ y(0) = 0, y'(0) = 3 \end{cases}$

Solution: a) $y(x) = -\frac{2}{3}e^{-2x} - \frac{1}{3}e^{x}$ b) $y(x) = \frac{1}{2}\sin(2x)e^{-x}$ c) $y(x) = -(6 + x)e^{-x} + x^2 - 4x + 6$ d) $y(x) = \frac{2\pi+1}{4}\cos 2x + \frac{1}{2}\sin 2x + x + \frac{1}{4}$ e) $y(x) = 2\sin(\frac{x}{3})$ f) $y(x) = (-1 + 2x)e^{x}$ g) $y(x) = e^{x}\left(\frac{3}{25}\cos 3x + \frac{62}{75}\sin 3x\right) + x^2 + \frac{2}{5}x - \frac{3}{25}$.

6.6. Solve the following boundary value problems.

$a)\begin{cases} y'' - 6y' + 9y = e^{3x} \\ y(0) = 0,\ y(1) = 0 \end{cases}$ $b)\begin{cases} y'' + 4y = 0 \\ y(0) = 0,\ y(\pi) = 0 \end{cases}$ $c)\begin{cases} y'' - 10y' + 25y = 50 \\ y(0) = 0,\ y(2) = 2. \end{cases}$

$d)\begin{cases} y'' - 8y' + 16y = 0 \\ y(0) = 1,\ y(1) = e^4. \end{cases}$

Solution: a) $y(x) = (-\frac{1}{2}x + \frac{1}{2}x^2)e^{3x}$ b) $y(x) = C\sin 2x$ c) $y(x) = (-2 + x)e^{5x} + 2$ d) $y(x) = e^{4x}$.

6.7. Knowing that $y = e^{2x}$ is a solution of the differential equation

$$y'' - \alpha y' + 10y = 0, \quad \alpha \in \mathbb{R},$$

determine α and the general equation of this equation.

Solution: $\alpha = 7$; $y(x) = C_1e^{2x} + C_2e^{5x}$.

6.8. Knowing that $y(x) = xe^{2x}$ is a solution of the differential equation $2y'' - \alpha y' + 8y = 0$, com $\alpha \in \mathbb{R}$, solve the boundary value problem $\begin{cases} 2y'' - \alpha y' + 8y = 16 \\ y(0) = 1;\ y(1) = 2 \end{cases}$.

Solution: $y(x) = -e^{2x} + xe^{2x} + 2$.

6.9. solve the following problem with periodic conditions.

$$\begin{cases} y'' + 4y = 4, \\ y(0) = y\left(\frac{\pi}{2}\right), y'(0) = y'\left(\frac{\pi}{2}\right) \end{cases}.$$

Solution: $y(x) = 1$.

6.10. Determine the general solution of the following differential equations.

$a) y' + y^2 \sin x = 0$ $\quad b) yy' + x = 0$ $\quad c) y'' - 2y' = 0$

$d) y'y - x(2y^2+1)e^{x^2} = 0$ $\quad e) \frac{dy}{dx} \cos y = -x\frac{\sin y}{1+x^2}$ $\quad f) y' + 6yx^5 - x^5 = 0$

Solution: a) $y^{-1}(x) = -\cos x + C$ b) $y^2(x) = -x^2 + C$ c) $y(x) = C_1 + C_2 e^{2x}$ d) $\frac{1}{4}\ln(2y^2+1) - \frac{1}{2}e^{x^2} = C$ e) $\ln|\sin y| + \frac{1}{2}\ln(1+x^2) = C$ f) $y(x) = \frac{1}{6} + Ce^{-x^6}$.

6.11. Determine the values of a and b for which e^{2x} and e^{-2x} are solutions to the differential equation $y'' + ay' + by = 0$. For those vales of a and b, compute the general solution to the differential equation.

Solution: $a = 0$ e $b = -4$; $y_h(x) = Ae^{2x} + Be^{-2x}, A, B \in \mathbb{R}$.

6.12. [Malthus populational growth]

(a) Compute the time evolution of a population level $y(t)$ knowing that: i) at each time t, the growth rate of the polulation , $\dfrac{dy/y}{dt}$, is equal to r (with $r > 0$); ii) at time $t = 0$ there are y_0 (millions of individuals).

(b) Estimate the value of the Portuguese population in the year 2020, knowing that in the year 2013 (by hypothesis $t = 0$) there is record of $y = 10.457$ (million individuals) and that $r = -0.00476$.

(c) Comment on the Malthusian hypothesis, by studying $\lim_{t\to\infty} y(t)$.

Solution:

a) $y(t) = y_0 e^{rt}$; b) $y(8) \approx 10.0663$; c) $+\infty$ if $r > 0$, y_0 if $r = 0$ and 0 if $r < 0$.

6.13. [Populational growth according to Verhulst]

(a) Compute the time evolution of a population level $y(t)$ knowing that: i) at each time t, the rate of growth of the population, $\frac{dy/y}{dt}$, is equal to r minus ay (r: natural growth rate; a: migratory or death rate; with $r > 0$ e $a > 0$); ii) at time $t = 0$ there is a record of y_0 million individuals.

(b) b)Estimate the value of the Portuguese population in the year 2020, knowing that in the year 2013 (by hypothesis $t = 0$) there is a record of $y = 10.457$ million individuals and that $r = -0.00229$, $a = 0.00033$.

c) Study $\lim_{t\to\infty} y(t)$.

6.14. [Domar's growth model]

Consider an economical model where i) the aggregated demand y_d, varies on time according to the equation $\frac{dy_d}{dt} = \frac{dI}{dt}\frac{1}{s}$, where $I = I(t)$ is the investment and s the marginal propensity to saving ($1/s$ is keynesiano multiplier); ii) the productive capacity ($y_c = \rho K$ - note: the productive capacity depends only on the stock of capital) follows the differential equation $\frac{dy_c}{dt} = \rho\frac{dK}{dt}$, where $K = K(t)$ is the *stock* of capital of the economy (naturally, $\frac{dK}{dt} = I(t)$). Obtain the time trajectory of the investment, $I(t)$, satisfying the equilibrium of Domar's model - the variation of the aggregated demand = variation of the productive capacity.

6.15. Consider the demand and supply functions of a given commodity, $Q_d = a - bP$; $Q_s = -c + dP$.

(a) Determine the time evolution of the price level $P(t)$ knowing that a each time t, the variation rate of $P(t)$ is proportional to the excess demand, i.e., $\frac{dP}{dt} = \alpha(Q_d - Q_s)$ and that $P(0) = P_0 \neq P_e = (a + c)/(b + d)$ (equilibrium price).

(b) Verify under which conditions we have $\lim_{t\to\infty} P(t) = P_e$.