

# Historical VaR for bonds - a new approach

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## ABSTRACT

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Bonds historical returns cannot be used directly to compute VaR because the maturities of returns implied by the historical prices do not have the relevant maturities to compute VaR. Given the so-called pull-to-par in bonds, with return volatilities necessarily decreasing with diminishing time-to-maturity, direct use of historical returns would lead to overestimation of the true VaR.

Market practice deals with the problem of computing VaR for portfolios of bonds or mixed portfolios with cumbersome methods of cashflow mappings.

In this paper we propose a new approach. We develop a technique to adjust bonds historical prices and extract from them an adjusted history of returns, that can be used directly to compute historical VaR for bonds or bond portfolios.

We illustrate the method using one concrete traded market zero-coupon bond, but the simplicity of the method makes this enough to the reader to understand how it would work with coupon bonds, portfolios of bonds or mixed portfolios.

# 1 Introduction

Despite all criticisms (see for instance [5]), historical simulation is still by far the most popular VaR method for most securities, as argued in [4].

It is well known that VaR computation, by historical simulation, of bond portfolios differs in important ways from VaR computation of stock portfolios [2]. Essentially, this is because the market historical prices of bonds imply returns with a variety maturities, none of them being the relevant maturity for the purpose of VaR computations.

Consider, for instance, daily returns on a bond that has lived 5 years but has still 2 years to maturity. The past daily returns have maturities from 7 to 2 years: we have a different maturity for each daily return we observe. However the returns we are interested in simulating are the one associated with the next day(s), whose maturity we have no history about.

This moves away the possibility of using market bonds historical returns, *directly* in VaR computations.

The usual strategy to deal with this problem is via collecting historical information on the entire term structure of interest rates (bootstrapping, interpolating, choosing a model, etc.) and then observe the evolution in time of the exact cash-flows maturities, at the time we want to compute VaR. Then one would still need to deal with aspects such as how to combine the VaR of the various, cash-flows (bucketing methods, etc). All this, to consider interest rate risk alone. To include credit risk, correlation risk, etc many more assumptions would have to be included using lots of different historical information sources. There are books that dedicate many chapters to this matter, and the authors themselves admit the methods are quite complex. For an overview of existing methods see [1].

This type of cash flow mapping methods by being subjective, and using lots of information sources, ruin the objectivity and the simplicity underlying the ideas of the VaR historical simulation. In the end of day, computations are based upon estimations made using informations from other bonds and *not* historical prices of the bonds that actually belong to the portfolio we want to compute the VaR for.

Here we take a different approach and show the price history of the bonds we are interested in can *indirectly* be used to do all the necessary computations. The method relies on one crucial assumption: mainly, that the yield-to-maturity (YTM) of a particular bond is a “good enough” measure of return that already takes into account “all” relevant risks (interest rate,

credit, liquidity, etc) for that bond. To us this seems easier to accept than accepting all assumptions, models, transition rating matrices, etc. needed to implement the market practice method are correct.

Given historical prices (and thus YTM) on any bond, the proposed method relies on adjusting for the “pull-to-par effect” and obtaining an history of *pulled prices*, based upon the historical YTM, but with the maturity relevant for the VaR computations.

Finally, when used for portfolios, the developed method strongly preserves the market implicit correlations between the instruments in the portfolio.

The remaining of the paper is organized as follows. In Section 2 we present how to adjust bond return series for the purpose of VaR computations. Section 3 illustrates the proposed method using real market quotes for a particular zero coupon bond. In Section 4.1 we show how to extend the method to coupon bonds an portfolios of bonds and discuss other usages of the pulling technique. Finally, Section 5 concludes the paper summarizing the main contributions to the literature. .

## 2 The Method

Consider the classical VaR computation at day  $n_{VaR}$ , with time horizon  $N$  days, and confidence level  $\alpha$  percent. For the sake of the argument and to introduce notation, suppose we would be interested in computing the VaR of a bond in the same way we do it for stocks.

Following the classical approach<sup>1</sup>, the  $N$ -days historically observed passed returns should be used to compute VaR at date  $n_{VaR} + N$ .

For simplicity, we consider a particular ZCB with maturity  $T > n_{VaR} + N$  and principal  $P$ . See the time line in Figure 1 for a graphical representation of these instants.

Let us denote by  $p(n, T)$ , the historical price of the ZCB at day  $n$ , for  $N < n \leq n_{VaR} < T$ , and by  $HR(n, N)$  the  $N$ -days historical return at day  $n$ , defined as in [3].

Then the  $HR(n, N)$ , possibly overlapping, historical gross returns are given by:

$$HR(n, N) = \frac{p(n, T)}{p(n - N, T)}, \quad n = N + 1, \dots, n_{VaR}, \quad (1)$$

<sup>1</sup>VaR historical simulation method is referred by some authors as non-parametric VaR.

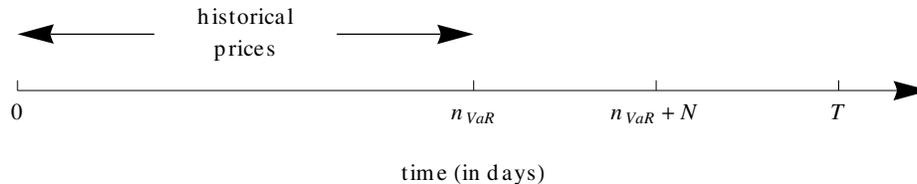


Figure 1: VaR computation time line.

and when be applied to the bond market value at day  $n_{VaR}$ ,

$$p(n_{VaR}, T) HR(n, N) = p(n_{VaR}, T) \frac{p(n, T)}{p(n - N, T)}, \quad n = N + 1, \dots, n_{VaR}, \quad (2)$$

would define an empirical distribution of possible  $N$ -days gross returns at time  $n_{VaR}$ .

The VaR would then be the potential loss of the  $1 - \alpha$  quantile of this empirical distribution. Figure 1 illustrates the idea underlying classical historical simulation VaR.

The above approach is naive and “suffers” from at various problems, when we are dealing with bonds:

- The most referred in the literature is that the historical price sequence  $p(n, T)$ , for  $n < n_{VaR}$ , used in Equation (2), implies a sequence of  $N$ -days gross returns associated with a different maturities of the bond  $T, T - 1, \dots, T - n_{VaR}$ . So, unless we assume the maturities of the bonds do *not* evolve over time the distribution of returns we are interested in is obviously *not* that of the returns in (2).
- Another problem, is that when we compute the VaR based on information up to  $n_{VaR}$ , we really want to access the risk at  $n_{VaR} + N$ . In

the case of bonds this is yet another maturity and, actually, the one we really want to access the risk at.

- Besides the above mentioned shortcomings, the classical method also suffers from all the problems associated the historical VaR itself, in particular, front the fact it uses the distribution of past returns to access future risk.

What one would like to know it would be the price  $p^*(n, T - n_{VaR})$  and  $p^*(n, T - n_{VaR} - N)$ , for all  $n$  with  $n < n_{VaR} < n_{VaR} + N$ . That is, we would like have access to historical prices on *our* ZCB with *fixed* maturities  $T - n_{VaR}$  and  $T - n_{VaR} - N$ . This information *does not exist in the market* and needs to somehow to be estimated.

## 2.1 Existing Methods

Market practice focus on trying to get  $p^*(n, T - n_{VaR})$  and  $p^*(n, T - n_{VaR} - N)$ , estimating the daily evolution of the *entire* term structure of interest rates. This requires quite an effort and relies on a variety of bonds different from the one we are interested in (many of them possibly not even in our portfolio), interpolation methods, bootstrapping, adjusting for credit, liquidity and other risk, and strong model assumptions.

Theoretically, the methods allows us to obtain ZCB artificial prices  $p^*(t, T)$  for all possible  $t$  and  $T$ . It would not be surprising if, after so many operations, the historical artificial prices of our concrete ZCB would differ from the true historical prices. If it is doubtful the method would work for our particular ZCB, it becomes even more questionable for big cash-flow matrices coming coupon paying bonds or bond' portfolios. In that case one would also need cash-flow mapping and bucketing methods.

## 2.2 Our approach

In this paper we propose a very different approach. Instead of focusing on the history of the *entire* yield curve, whose exact meaning is questionable. After all each bond has its own YTM, and there is no such thing as a yield curve for a particular bonds.<sup>2</sup>

<sup>2</sup>For a particular bond there is only one YTM, for one (different) maturity each day.

Our proposal is to consider each bond *individually* doing the necessary maturity adjustments to obtain the necessary returns. One would, then, compute the historical VaR for each bond, in a similar way one does for stocks. In our view this has the advantages of being simple, in line with the spirit of historical simulation VaR, and of using information actually available in the market and related to the particular bond(s) in our portfolio. The idea is intuitively quite simple – we adjust past prices for the “pull-to-par” effect!

For all  $n < m$  we can define the *pulled price*  $f(m, n, T)$  as the price pulled to time  $m$ , considering the YTM observed at time  $n$ , of our ZCB  $T$ -bond.

Based upon historical prices  $p(n, T)$ , we can define the implied daily compounded yield<sup>3</sup>  $r(n, T - n)$ ,

$$r(n, T - n) = \left( \frac{P}{p(n, T)} \right)^{\frac{1}{T-n}} - 1. \quad (3)$$

We now define  $f(m, n, T)$ , the bond price pulled to time  $m$  and fixed by the price  $p(n, T)$  (at time  $n$ ) for  $n < m < T$ , as :

$$f(m, n, T) = \frac{P}{(1 + r(n, T - n))^{T-m}} = \frac{P}{\left( \frac{P}{p(n, T)} \right)^{\frac{T-m}{T-n}}}. \quad (4)$$

The relevant maturities for computing an  $N$ -day VaR at time  $n_{VaR}$  are  $T - n_{VaR}$  and  $T - (n_{VaR} + N)$ . Thus, will be interested in using past yields from time  $n$ ,  $n < n_{VaR}$ , to obtain prices pulled to the times  $n_{VaR}$  and  $n_{VaR} + N$ .

We can now define the  $n_{VaR}$  time to maturity *adjusted*<sup>4</sup>  $N$ -days historical gross return at day  $n$ , denoted by  $AHR(n, N, n_{VaR})$ , as the quotient between  $f(n_{VaR} + N, n, T)$  and  $f(n_{VaR}, n - N, T)$ :

$$AHR(n, N, n_{VaR}) = \frac{f(n_{VaR} + N, n, T)}{f(n_{VaR}, n - N, T)}, \quad n = N + 1, \dots, n_{VaR}. \quad (5)$$

Substituting Equation (4) in Equation (5), we can rewrite the adjusted historical gross returns and obtain them directly from historical (market

<sup>3</sup>Daily compounded yields are used because typical VaR time horizons are specified in days. Alternative compounding could be used, but it does make sense to match the compounding with the VaR horizon.

<sup>4</sup>Note adjusted returns are computed using pulled prices  $f(m, n, T)$  and not historical prices  $p(n, T)$ .

observed) prices:

$$AHR(n, N, n_{VaR}) = \frac{\left(\frac{P}{p(n-N, T)}\right)^{\frac{T-n_{VaR}}{T-(n-N)}}}{\left(\frac{P}{p(n, T)}\right)^{\frac{T-(n_{VaR}+N)}{T-n}}}, \quad n = N + 1, \dots, n_{VaR}. \quad (6)$$

Note that the  $AHR(n, N, n_{VaR})$  value is fixed by historical market prices  $p(n, T)$  and  $p(n - N, T)$ , thus also capturing the pull-to-par effect and the market changes in between  $n - N$  and  $n$ , while being adjusted to the VaR computation relevant maturities, namely,  $T - n_{VaR}$  and  $T - (n_{VaR} + N)$ .

Our proposal is to replace the original historical returns of Equation (1) by those of Equation (6). Using these adjusted historical returns directly in the VaR computation, the VaR is the potential loss of the  $1 - \alpha$  quantile of the following time to maturity adjusted empirical distribution:

$$p(n_{VaR}, T)AHR(n, N, n_{VaR}) = p(n_{VaR}, T) \frac{f(n_{VaR} + N, n, T)}{f(n_{VaR}, n - N, T)} \quad (7)$$

for  $n = N + 1, \dots, n_{VaR}$ .

### 3 Illustration

In this section we illustrate the method using market data from a particular zero coupon corporate bond,  $B$ . The prices were obtained from a quote service that delivers market prices aggregated from different dealers responsible for trading (market makers) this particular bond. Our bond  $B$  has a principal of  $P = 1000$  and matures at day  $T = 731$ . We are standing at day  $n = 372$  (today) and will be interested in computing the 10-day VaR of this particular bond ( $N = 10$ ). Figure 2 shows, in percentage, the real market historical prices. The prices in Figure 2 imply the evolution of yields in Figure 3. Recall from Figure 3 that each day the YTM corresponds to a different time to maturity, marked in the graph top axis. Figure 3 clearly shows the usual trend observed in the market, in which smaller time to maturities are traded with smaller implied returns.

#### 3.1 Adjusting one return

For illustration purposes, consider we want to compute one single adjusted return for a VaR computation with  $N = 10$  at time  $n_{VaR} = 372$ . In particular

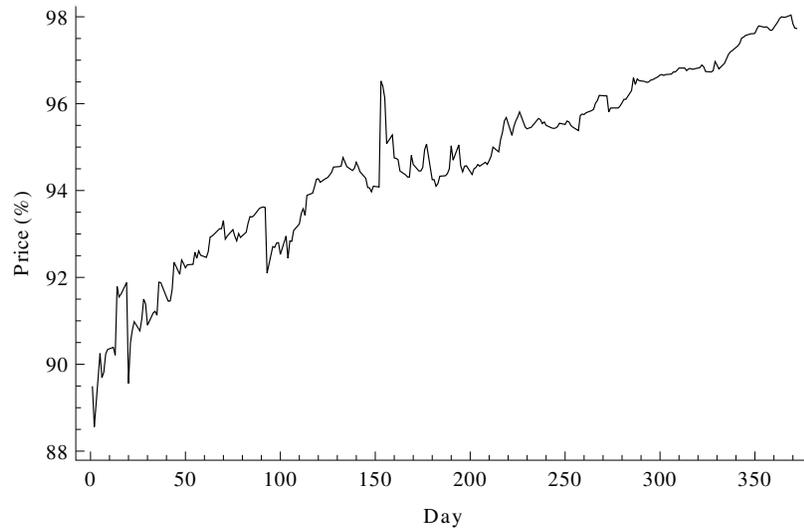


Figure 2: Real historical prices of a zero coupon bond with principal  $P = 1000$  and maturing at day  $T = 731$ , as a percentage of the principal.

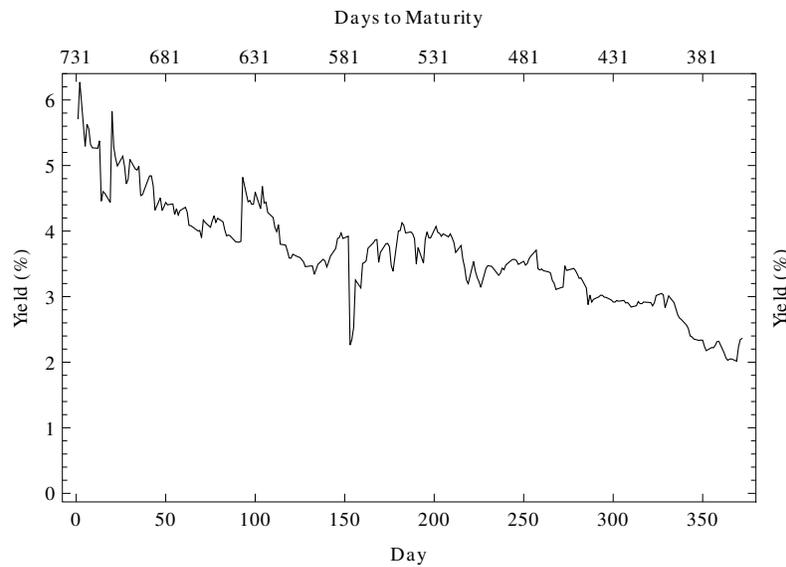


Figure 3: Daily compounded yields, implied by the historical prices of Figure 2, as a function of both time and time to maturity.

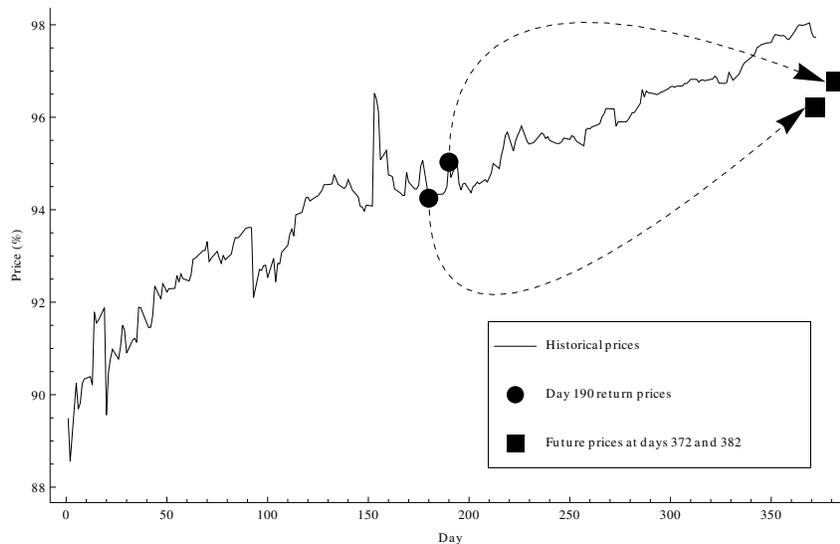


Figure 4: The prices that determine the  $N = 10$  days historical return at time  $n = 190$ , the corresponding future prices at times  $n_{VaR} = 372$  and  $n_{VaR} + N = 372 + 10 = 382$ , along with the historical prices sequence. The arrows represent future values.

we want to adjusted the historical return observed on  $n = 190$ , i.e we want to compute is  $AHR(190, 10, 372)$ . We observe the historical prices  $p(180, 731)$  and  $p(190, 731)$ , associated with maturities  $731 - 180 = 551$  and  $731 - 190 = 541$ , respectively. But we would like to have the pulled prices for the VaR relevant maturities, i.e., for the maturities  $T - n_{VaR} = 731 - 372 = 359$  and  $T - n_{VaR} + N = 731 - 372 + 10 = 349$ .

Table 1 shows the market observed historical return at day  $n = 190$ , as well as the corresponding adjusted return for VaR computed at day  $n_{VaR} = 372$ . The annualized daily yields to maturity and future prices used to compute the adjusted return are also detailed.

As it can be observed from Table 1 the adjusted return is closer to one than the historical return. This is in accordance with the trend observed in Figure 3 and the annualized YTM computed in Table 1 . The prices that determine this historical return are highlighted in Figure 4 with the black circles. While the pulled ones that determine adjusted historical returns are represented with black squares.

Historical gross return $HR$		
$p(190 - 10, 731)$	$p(190, 731)$	$HR(190, 10) = \frac{p(190,731)}{p(180,731)}$
94.25	95.03	<b>1.00828</b>

(a)

Annualized daily YTM (%)	
$r(180, 551)$	$r(190, 541)$
4.001	3.499

(b)

Adjusted gross return $AHR$		
$f(372, 190 - 10, 731)$	$f(372 + 10, 190, 731)$	$AHR(190, 10, 372) = \frac{f(382,190,731)}{f(372,180,731)}$
96.215	96.765	<b>1.00571</b>

(c)

Table 1: (a)  $N = 10$  days market observed historical return at day  $n = 190$ . (b) Days  $n = 180$  and  $n = 190$  annualized daily yields to maturity. (c) Adjusted  $N = 10$  days return at day  $n = 190$ , for VaR computed at day  $n_{VaR} = 372$ .

Time horizon	Confidence level	VaR	Quantile	Correlation
$N = 10$	$\alpha = 99\%$	-9.576	-9.935	0.984

Table 2: Time horizon  $N = 10$ , confidence level  $\alpha = 99\%$ , bond  $B$  VaR, computed at day  $n_{VaR} = 372$  by historical simulation using adjusted historical returns.

### 3.2 Computing VaR

Consider the VaR, with a time horizon of  $N = 10$  days and confidence level  $\alpha = 99\%$ , computed by historical simulation, at day  $n_{VaR} = 372$ , of the portfolio containing the bond  $B$ .

We want to obtain the entire empirical distribution of all possible 10 days returns. The maturities we are interested in are  $731 - 372 = 359$  and  $731 - 382 = 349$  days to maturity. In order to obtain this distribution we adjust each of the historical returns with Equation (6). The VaR is, then, computed using the empirical distribution of the adjusted returns of Equation (6). In order to obtain this distribution the adjustment of the single return detailed in the previous section is repeated for all available historical returns. Figure 5 shows the real, market observed historical prices, and also, the corresponding pulled prices,  $f(m, n, T)$  of Equation (4), to days  $m = n_{VaR} = 372$  and  $m = n_{VaR} + N = 372 + 10 = 382$ . The pulled prices are plotted as a function of the day  $n$  of the historical price  $p(n, T)$  which fixes the yield value  $f(m, n, T)$ . The prices highlighted in Figure 4 with the black circles and squares are highlighted again in Figure 5, but now plotted as a function of  $n$ .

Figure 6 shows the sequence of the bond's historical returns along with the sequence of the corresponding adjusted returns computed from the fictitious prices of Figure 5. Figure 7 shows the respective histograms.

Finally, Table 2 shows the VaR value computed from the empirical distribution of the overlapping adjusted returns of Equation (7), along with the possible loss corresponding to  $1 - \alpha$  quantile of the overlapping historical returns empirical distribution of Equation (2), for comparison purposes. It also shows the correlation coefficient between the original and the adjusted returns.

As it can be observed from Figure 6, the adjusted returns are closer to one than the historical ones. This can be observed as well in Figure 7 where the adjusted returns histogram is more concentrated towards one than the

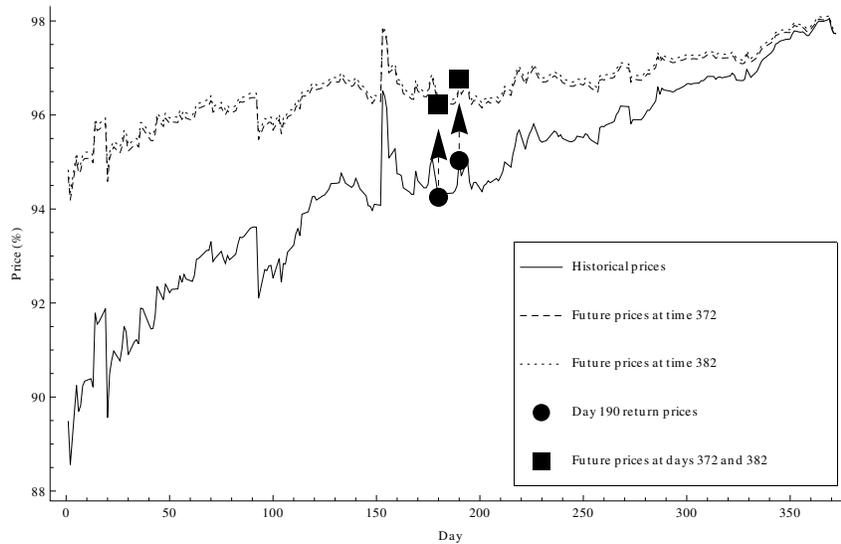


Figure 5: Historical prices, and the corresponding pulled prices at times  $m = n_{VaR} = 372$  and  $m = n_{VaR} + N = 372 + 10 = 382$ . The pulled prices are plotted as a function of the time  $n$ , of the historical price  $p(n)$ , that fixed the future price.

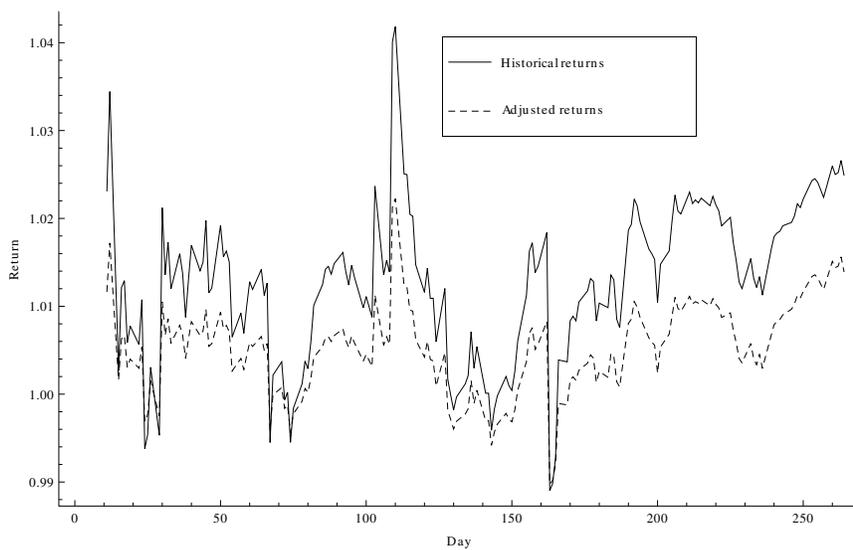


Figure 6: Historical returns and the corresponding adjusted returns for  $n_{VaR} = 372$ .

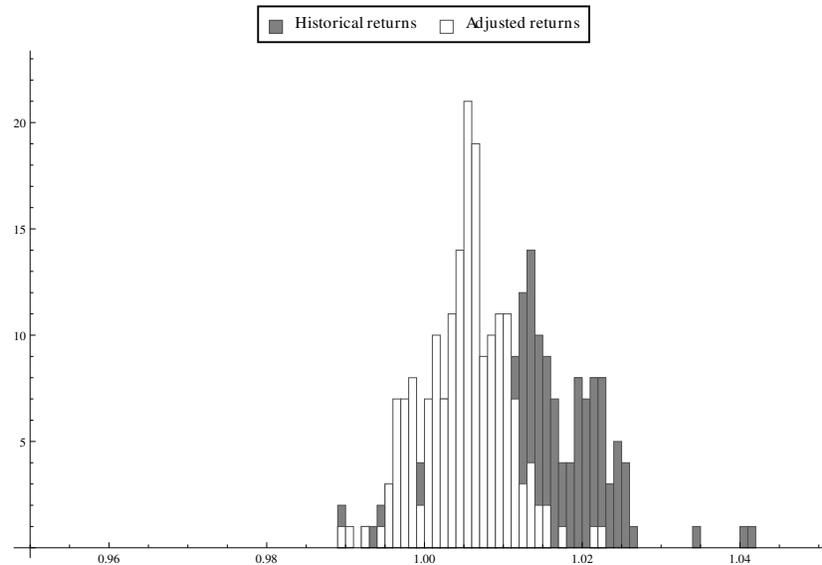


Figure 7: Historical returns and the corresponding adjusted for  $n_{VaR} = 372$  returns, histograms.

historical returns histogram. This results in a VaR value smaller than the corresponding loss of the  $1 - \alpha/100$  quantile of the possible historical returns. Again, this result conforms with Figure 3 which shows a clear decreasing trend in interest rate as time to maturity decreases.

## 4 Other usages and Extensions

In this section we show how the “pulling of prices” proposed in the previous section can be also useful for other purposes – like inferring from the history of other bonds, history on just issued bonds – and we discuss the extension VaR method developed to coupon bonds and bond portfolios.

### 4.1 Adjusting for past values

Suppose that the bond  $B$  has already expired and its issuer issues a new bond,  $B_1$ , similar to bond  $B$ . I.e., with the same type, principal, maturity, number of coupons, coupon rate (if applicable), etc.

Consider a portfolio that contains the bond  $B_1$ . The portfolio VaR with

Time horizon	Confidence level	VaR	Quantile	Correlation
$N = 10$	$\alpha = 99\%$	-20.291	-9.935	0.985

Table 3: Time horizon  $N = 10$ , confidence level  $\alpha = 99\%$ , bond  $B$  VaR, computed at day  $n_{VaR} = 1$  by historical simulation using adjusted historical returns.

time horizon  $N$  days is to be computed by historical simulation at day 1. The only historical prices available from bond's  $B_1$  issuer are those of bond  $B$ .

The adjustment method proposed in this paper can still be applied to pull the historical prices of bond  $B$  to past times, times before the historical prices were observed, namely, for day 1. The process is the same as for future dates: for each historical price compute the daily yield to maturity implied by the historical price; than compute the bond's value at a previous time valuing the bond's cash flows following the time considered, with the implied daily yield to maturity; finally use the previous times values to get the adjusted returns of Equation (5), and compute the VaR.

Consider the portfolio containing the bond  $B_1$  and the VaR computed by historical simulation at day  $n = 1$  with the historical prices of bond  $B$ , showed in Figure 2.

Following section 4.1 the past values of Equation (4),  $f(m, n)$  with  $m = 1 \leq n$  are used to compute the adjusted historical returns of Equation (6) and the vaR is computed from the resulting empirical distribution. In this section we illustrate this process by repeating the figures and the table of the previous section, but now, for  $n_{VaR} = 1$ .

It can be observed from Figure 11 that the adjusted returns are now less concentrated towards one than the historical returns. This results in a VaR value, showed in Table 3, which is now greater than the corresponding loss of the  $1 - \alpha/100$  quantile of the possible historical returns. Again, this observation in accordance with Figure 3 and the fact that the time to maturity at time VaR is computed,  $n_{VaR} = 1$ , is greater that the times to maturity at following times, namely, when the historical prices were observed.

## 4.2 Coupon bonds

The extension to coupon bonds is straight forward. In order to compute the pulled price of a coupon bond at time  $m$ , based on the market price of the

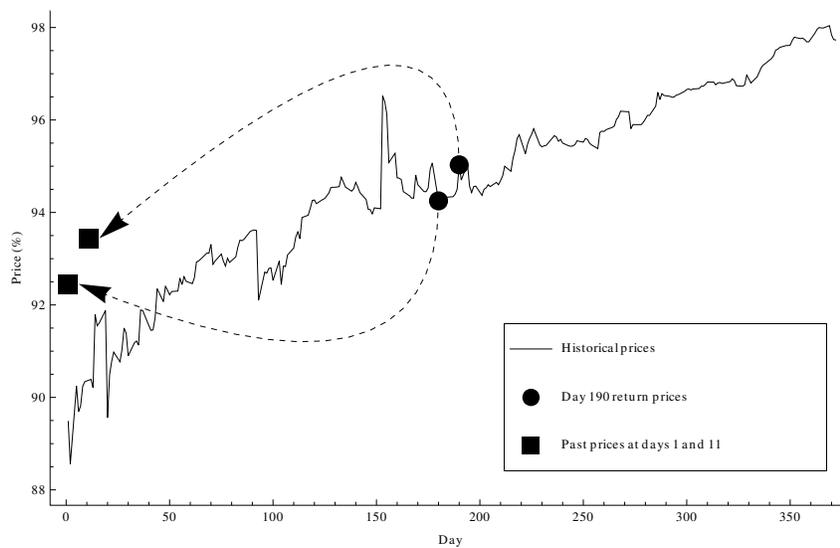


Figure 8: The prices that determine the  $N = 10$  days historical return at time  $n = 190$ , the corresponding past prices at times  $n_{VaR} = 1$  and  $n_{VaR} + N = 1 + 10 = 11$ , along with the historical prices sequence. The arrows represent past values.

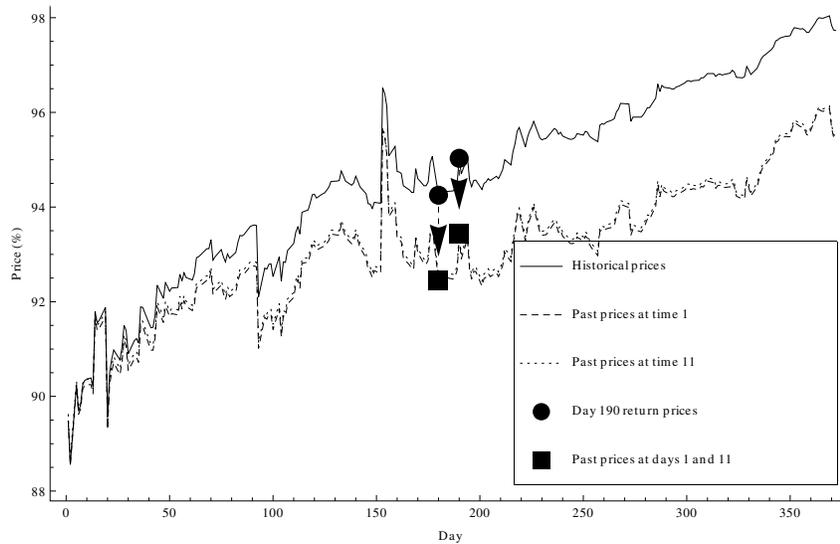


Figure 9: Historical prices, and the corresponding past prices at times  $m = n_{VaR} = 1$  and  $m = n_{VaR} + N = 1 + 10 = 11$ . The past prices are plotted as a function of the time  $n$ , of the historical price  $p(n)$ , that fixed the future price.

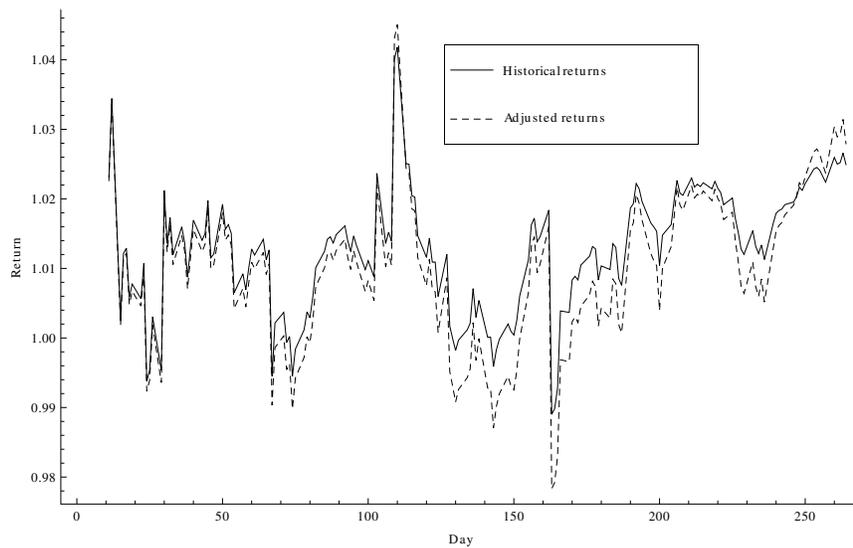


Figure 10: Historical returns and the corresponding adjusted returns for  $n_{VaR} = 1$ .

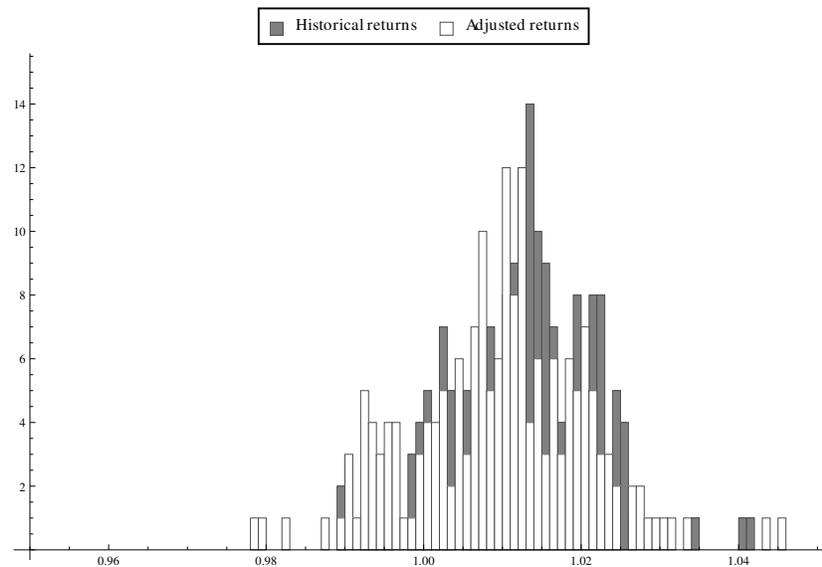


Figure 11: Historical returns and the corresponding adjusted for  $n_{VaR} = 1$  returns, histograms.

bond at time  $n < m$ , two differences from the zero coupon bond case arise:

- the yield to maturity at time  $n$  is computed using the bond's dirty price and accounting for all future cash flows after time  $n$ ;
- the value of the bond at time  $m$  accounts for all future cash flows after time  $m$ .

Then, the adjusted returns are defined by Equation (6) as in the case of a zero coupon bond and the VaR is computed as the loss corresponding to the quantile of the empirical distribution of Equation (7).

Using the proposed approach dealing with portfolios of bonds or mixed portfolios should be not harder than dealing with stock portfolios, provided we always consider the adjusted returns history and not the actual returns.

## 5 Conclusions

Bond historical returns can not be used directly to compute VaR by historical simulation because the maturities of the yields implied by the historical prices are not the relevant maturities at time VaR is to be computed.

In this paper we adjust bonds historical returns so that the adjusted returns can be used directly to compute VaR by historical simulation. The adjustment is based on pulled prices, extracted from historical prices, and pulled to the maturities relevant for the VaR computation.

The proposed method has the following features:

- The time to maturity adjusted bond returns are used directly in the VaR historical simulation computation.
- VaR of portfolios with bonds can be computed by historical simulation keeping the simplicity of the historical simulation method.
- The portfolio specific VaR is obtained.
- The VaR values obtained are consistent with the usual market trend of smaller times to maturity being traded with smaller interest rates, therefore carrying smaller risk and having a smaller VaR.
- The only source of information used is the market, through the bonds historical prices.
- The correlation between each bond return and the returns of the other instruments in the portfolio is strongly preserved.
- The VaR for the desired time horizon is computed directly with no VaR time scaling approximations.

We left for future work, the research of the mathematical properties of the developed method, and backtesting the method with benchmark portfolios.

## References

- [1] C. Alexander. *Market Risk Analysis, Value at Risk Models*. John Wiley & Sons, 2009.
- [2] Gangadhar Darbha. Value-at-risk for fixed income portfolios a comparison of alternative models. Technical report, National Stock Exchange, Mumbai, 2001.
- [3] John C. Hull. *Options, Futures, and Other Derivatives*. Prentice Hall, 7 edition, 2008.

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- [4] Christophe Pérignon and Daniel R. Smith. The level and quality of value-at-risk disclosure by commercial banks. *Journal of Banking & Finance*, 34(2):362 – 377, 2010.
  - [5] Matthew Pritsker. The hidden dangers of historical simulation. *Journal of Banking & Finance*, 30(2):561–582, February 2006.