Mathematics admission exam

International students

Solution Topics

Date: 06/06/2023 **Duration:** 180 min

Part 1

1. The access password to a certain internet service is made of six characters. Two of those characters must be digits (0, 1, 2, 3, 4, 5, 6, 7, 8 or 9) and the remaining four characters must be uppercase vowels (A, E, I, O or U). For instance, AA2E2U and U1IEA8 are valid passwords. How many valid passwords can be formed in this way?

(A) 937500 (B) 318750 (C) 1875000 (D) 468750

2. The sum of all elements in a certain line in Pascal's triangle is 4096. What is the fourth element in that line?

(A) 286 (B) 495 (C) 220 (D) 715

3. Consider a function defined in $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ through the expression $f(x) = 2\log_5\left(\frac{x^3}{25}\right)$. Which of the following expressions can also be used to define f?

(A) $3\log_5(x)$ (B) $6\log_5(x) - 4$ (C) $6\log_5\left(\frac{x}{25}\right)$ (D) $6\log_5(x) - 50$

4. Consider a function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = \sin(e^{3x})$. Which of the following expressions defines the derivative of f?

(A) $\cos(3e^{3x})$ (B) $3e^{3x}\cos(3e^{3x})$ (C) $3\cos(e^{3x})$ (D) $3e^{3x}\cos(e^{3x})$

5. Consider a function $f : \mathbb{R}^+ \to \mathbb{R}$, defined by $f(x) = \ln(3x)$, as well as a point M, corresponding to the intersection of the graph of f with the horizontal axis. Which of the following equations defines the tangent to the graph of f at point M?

(A) $y = x - \frac{1}{3}$ (B) $y = 6x - 2$ (C) $y = 6x - 3$ (D) $y = 3x - 1$

6. How many complex solutions of the equation $z^9 = 2$ have their geometric images in the second quadrant?

(A) None (B) Two (C) Three (D) Four

7. Consider a function f with domain $\mathbb{R}$ and image $[-3, 5]$. What is the image of function g, defined in $\mathbb{R}$ through the expression $g(x) = |f(x-1)| + 3$?

(A) $[6, 8]$ (B) $[5, 7]$ (C) $[3, 5]$ (D) $\mathbf{[3, 8]}$

Part 2

1. Consider a complex number, z, such that $\mathrm{Re}(z) > 0$ and $\mathrm{Im}(z) = -\mathrm{Re}(z)$. To which quadrant in the complex plane belongs the geometrical image of z^{27}?

2. Let Ω be the space of events associated to a certain random experiment. Given an event $X \subset \Omega$, denote by $P(X)$ the probability of X and by $\bar{X}$ the complementary event.

Consider two events A, B such that $P(B) < 1$. Show that $P(\bar{B}) \neq 0$ and that

$$P(A \cap B) = P(A) - P(\bar{B}) + P(\bar{A}|\bar{B}) \times P(\bar{B}).$$

3. Suppose that five 1 euro coins and six fifty cents coins are randomly placed in a 4×4 checker board.

(a) What is the probability of one of the four horizontal lines being left with no coins?

(b) What is the probability of one of the diagonals being totally filled with coins having the same face value?

4. Consider function f, with domain $\mathbb{R}^+$, defined by the expression

$$f(x) = 3x - 2\ln x + \frac{1}{x}.$$

Using exclusively analytical methods:

(a) Study the behaviour of f regarding monotony and the existence and nature of local minima/maxima.

(b) Study function f regarding the regions where its graph is up or down concave.

(c) Study the existence of asymptotes to the graph of f.

(d) Draw the graph of f.

5. Let f be the function with domain $\mathbb{R} \setminus 0$ defined by

$$\begin{cases} \dfrac{7x}{e^{3x} - 1}, & \text{if } x > 0 \\ \dfrac{\sin(7x)}{3x}, & \text{if } x < 0 \end{cases}.$$

Answer the following questions using exclusively analytical methods.

(a) What is the value that must be assigned to $f(0)$ in order to extend f to $\mathbb{R}$ as a continuous function? Justify.

(b) Show, using the intermediate value theorem, that there exists $x_0 \in \left]-\pi, -\frac{\pi}{14}\right[$ such that $f(x_0) = 1$.

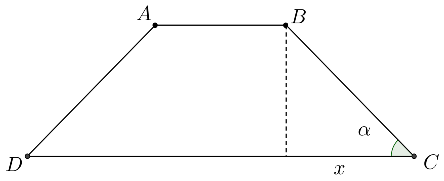
6. Consider a function f, defined on $[0, +\infty[$ by the expression $f(x) = \ln(3x + e^x)$.

(a) Compute $\lim_{x \to +\infty} f(x)$.

(b) Compute $\lim_{x \to +\infty} f'(x)$.

7. Consider an isosceles trapezium $[ABCD]$ with $\bar{BC} = \bar{AD} = 7$cm, $\bar{AB} < \bar{CD}$ and $\bar{AB} = 5$cm. Let $\alpha \in]0, \frac{\pi}{2}[$ be the amplitude, measured in radians, of the angle $D\hat{C}B$.

(a) Show that the area of $[ABCD]$ is given, in cm^2, by $A(\alpha) = 49 \sin\alpha \cos\alpha + 35 \sin\alpha$.



(b) Obtain an expression for $A'(\alpha)$.

(c) Knowing that $\tan\alpha = 5$, compute the exact value of $A(\alpha)$.

Scores

Part I

Question	1	2	3	4	5	6	7
Points	1	1	1	1	1	1	1

Part 2

Question	1	2	3a	3b	4a	4b	4c	4d	5a	5b	6a	6b	7a	7b	7c
Points	1	1	1	1	1	0.5	0.5	0.5	1	1	0.5	1	1	1	1

Lisbon School of Economics & Management
Universidade de Lisboa

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Part 1

The correct options are marked in green.

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Part 2

1. Consider a complex number, z, such that $\mathrm{Re}(z) > 0$ and $\mathrm{Im}(z) = -\mathrm{Re}(z)$. To which quadrant in the complex plane belongs the geometrical image of z^{27}?

Solution: The complex number z can be written in the form $z = \rho e^{-i\frac{\pi}{4}}, \rho > 0$, hence, $z^{27} = \rho^{27} e^{-i\frac{27\pi}{4}}$. Noting that $-\frac{27\pi}{4} = -7\pi + \frac{\pi}{4}$, we conclude that the geometrical image of z^{27} is in the third quadrant.

2. Let Ω be the space of events associated to a certain random experiment. Given an event $X \subset \Omega$, denote by $P(X)$ the probability of X and by $\bar{X}$ the complementary event.

Consider two events A, B such that $P(B) < 1$. Show that $P(\bar{B}) \neq 0$ and that

$$P(A \cap B) = P(A) - P(\bar{B}) + P(\bar{A}|\bar{B}) \times P(\bar{B}).$$

Solution: We have $P(\overline{B}) = 1 - P(B) > 1 - 1 = 0$ since $P(B) < 1$.
Furthermore,

$$P(A) - P(\overline{B}) + P(\overline{A}|\overline{B}) \times P(\overline{B})$$

$$= P(A) - P(\overline{B}) + P(\overline{A} \cap \overline{B}) \text{ [definition of conditional probability]}$$

$$= P(A) - P(\overline{B}) + P(\overline{A \cup B}) \text{ [De Morgan's rule]}$$

$$= P(A) - P(\overline{B}) + 1 - P(A \cup B) \text{ [probability of the complementary]}$$

$$= P(A) + P(B) - P(A \cup B) \text{ [probability of the complementary]}$$

$$= P(A \cap B).$$

3. Suppose that five 1 euro coins and six fifty cents coins are randomly placed in a 4×4 checker board.

(a) What is the probability of one of the four horizontal lines being left with no coins?

Solution: The number of possible configurations corresponds to distributing the five one euro coins in 16 available positions and then distributing the six fifty cents coins in the remaining 11 positions. Thus, the number of possible configurations is given by

$$\binom{16}{5} \times \binom{11}{6} = 2018016$$

The number of favourable configurations can be computed in the following way: i. choose which of the four lines will be empty; ii. place the five 1 euro coins in the 12 positions out of the empty line; iii. place the six 50 cents coins in the remaining seven positions. Thus, the number of favourable configurations is

$$\frac{4 \times \binom{12}{5} \times \binom{7}{6}}{\binom{16}{5} \times \binom{11}{6}} = \frac{1}{91}.$$

Note: $\binom{n}{p}$ represents the number of combinations of n objects in groups of p, that is

$$\binom{n}{p} = \frac{n!}{p!(n-p)!}$$

(b) What is the probability of one of the diagonals being totally filled with coins having the same face value?

Solution: The number of possible cases has already been discussed in the previous answer. Regarding the number of favourable cases, we can start by choosing which of the two diagonals will be filled with coins of equal value and, depending on whether they are one-euro coins or fifty-cent coins, arrange the remaining coins in the twelve remaining positions. It is then necessary to subtract the situations in which both diagonals are filled with coins of the same value. Specifically, the number of favourable cases is given by

$$2 \times \left(\binom{12}{1} \times \binom{11}{6} + \binom{12}{2} \times \binom{10}{5} - \binom{8}{1} \times \binom{7}{2} \right) = 44016,$$

and the requested probability is

$$\frac{2 \times \left(\binom{12}{1} \times \binom{11}{6} + \binom{12}{2} \times \binom{10}{5} - \binom{8}{1} \times \binom{7}{2} \right)}{\binom{16}{5} \times \binom{11}{6}} = \frac{44016}{2018016} = \frac{131}{6006}.$$

4. Consider function f, with domain $\mathbb{R}^+$, defined by the expression

$$f(x) = 3x - 2\ln x + \frac{1}{x}.$$

Using exclusively analytical methods:

(a) Study the behaviour of f regarding monotony and the existence and nature of local minima/maxima.

Solution: We have, for all $x > 0$,

$$f'(x) = 3 - \frac{2}{x} - \frac{1}{x^2} = \frac{3x^2 - 2x - 1}{x^2} = \frac{3(x-1)(x+\frac{1}{3})}{x^2}$$

since the roots of the second degree polynomial $P(x) = 3x^2 - 2x - 1$ are $x_1 = 1$ and $x_2 = -\frac{1}{3}$.
Hence,

- $f'(x) < 0$ for $x \in]0; 1[$;
- $f'(x) = 0$ for $x = 1$;
- $f'(x) > 0$ for $x \in]1; +\infty[$.

Finally, f is non-increasing in $]0; 1]$, non-decreasing in $[1; +\infty[$ and f achieves a global minimum at $x = 1$.

(b) Study function f regarding the regions where its graph is up or down concave.

Solution: For all $x > 0$,

$$f''(x) = \frac{2}{x^2} + \frac{2}{x^3} = \frac{2x+2}{x^3} > 0.$$

Hence the graph of f is concave upward.

(c) Study the existence of asymptotes to the graph of f.

Solution: We have $\lim_{x \to 0^+} f(x) = 3 \times 0 - 2(-\infty) + (+\infty) = +\infty$, hence $x = 0$ is a vertical asymptote to the graph of f.
We now look for asymptotes at $+\infty$:

$$\lim_{x \to +\infty} \frac{f(x)}{x} = 3 - 0 + 0 = 3,$$

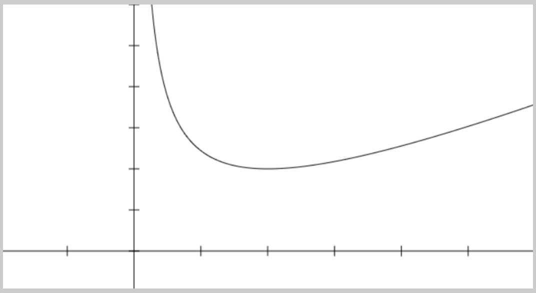
and

$$\lim_{x \to +\infty} f(x) - 3x = \lim_{x \to +\infty} -2\ln(x) + \frac{1}{x} = -\infty,$$

and the graph of f does not possess any asymptote at $+\infty$.

(d) Draw the graph of f.

Solution:



5. Let f be the function with domain $\mathbb{R} \setminus 0$ defined by

$$\begin{cases} \dfrac{7x}{e^{3x}-1}, & \text{if } x > 0 \\ \dfrac{\sin(7x)}{3x}, & \text{if } x < 0 \end{cases}.$$

Answer the following questions using exclusively analytical methods.

(a) What is the value that must be assigned to $f(0)$ in order to extend f to $\mathbb{R}$ as a continuous function? Justify.

Solution: Let's start by noting that the function is continuous for $x \neq 0$. In fact, when $x > 0$, the function is defined as the quotient of continuous functions (a polynomial and the sum of an exponential function with a constant), and the denominator does not equal zero (because $x \neq 0$), making it continuous. When $x < 0$, the function is also a quotient of continuous functions (sine and polynomial), and the denominator is never zero (because $x \neq 0$), so it is also continuous. Therefore, it will be possible to extend f continuously to the set $\mathbb{R}$ if and only if it can be made continuous at $x = 0$. Now, for f to be continuous at $x = 0$, it is necessary and sufficient that

$$f(0) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x).$$

Computing

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0} \frac{\sin(7x)}{3x} = \frac{7}{3} \cdot \underbrace{\lim_{x\to 0} \frac{\sin(7x)}{7x}}_{=1} = \frac{7}{3}$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0} \frac{7x}{e^{3x}-1} = \left(\frac{3}{7} \cdot \underbrace{\lim_{x\to 0} \frac{e^{3x}-1}{3x}}_{=1} \right)^{-1} = \frac{7}{3},$$

we conclude that if we define $f(0) = \frac{7}{3}$ the function f will be continuous in $\mathbb{R}$.

(b) Show, using the intermediate value theorem, that there exists $x_0 \in \left]-\pi, -\frac{\pi}{14}\right[$ such that $f(x_0) = 1$.

Solution: It has already been established in the previous question that the function f is continuous on the interval $\left]-\pi, -\frac{\pi}{14}\right[$. Since $f(-\pi) = 0$ and $f(-\frac{\pi}{14}) = \frac{14}{3\pi}$, the intermediate value theorem guarantees that, within this interval, the function f takes on all values between 0 and $\frac{14}{3\pi}$. In particular, since $0 < 1 < \frac{14}{3\pi}$, there exists at least one point $x_0 \in \left]-\pi, -\frac{\pi}{14}\right[$ such that $f(x_0) = 1$.

6. Consider a function f, defined on $[0, +\infty[$ by the expression $f(x) = \ln(3x + e^x)$.

(a) Compute $\lim\limits_{x \to +\infty} f(x)$.

Solution:

$\lim\limits_{x \to +\infty} f(x) = +\infty;$

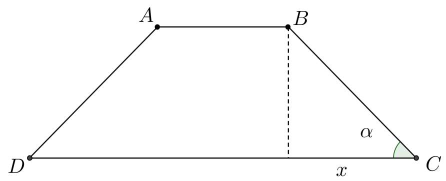
(b) Compute $\lim\limits_{x \to +\infty} f'(x)$.

Solution:

$$\lim_{x \to +\infty} f'(x) = \lim_{x \to +\infty} \frac{3 + e^x}{3x + e^x} = \lim_{x \to +\infty} \frac{\frac{3}{e^x} + 1}{\frac{3x}{e^x} + 1} = 1;$$

7. Consider an isosceles trapezium $[ABCD]$ with $\bar{BC} = \bar{AD} = 7\text{cm}$, $\bar{AB} < \bar{CD}$ and $\bar{AB} = 5\text{cm}$. Let $\alpha \in]0, \frac{\pi}{2}[$ be the amplitude, measured in radians, of the angle $D\hat{C}B$.

(a) Show that the area of $[ABCD]$ is given, in cm^2, by $A(\alpha) = 49 \sin\alpha \cos\alpha + 35 \sin\alpha$.



Solution: $A(\alpha) = \dfrac{\overline{CD} + \overline{AB}}{2} . h$, where h is the height of the trapezium.

As $\sin\alpha = \frac{h}{7}$ and $\cos\alpha = \frac{x}{7}$, we obtain $h = 7\sin\alpha$ e $x = 7\cos\alpha$;

So $\overline{AB} = 5$ and $\overline{CD} = 5 + 2x = 5 + 14\cos\alpha$;

Then,

$$A(\alpha) = \frac{10 + 14\cos\alpha}{2} . 7\sin\alpha = 35\sin\alpha + 49\sin\alpha\cos\alpha$$

(b) Obtain an expression for $A'(\alpha)$.

Solution:
$A'(\alpha) = 49\cos^2\alpha - 49\sin^2\alpha + 35\cos\alpha$

(c) Knowing that $\tan\alpha = 5$, compute the exact value of $A(\alpha)$.

Solution: We have $\tan^2\alpha + 1 = \frac{1}{\cos^2\alpha}$. As $\tan\alpha = 5$ we obtain

$$\cos\alpha = \frac{\sqrt{26}}{26} \quad \text{e} \quad \sin\alpha = \frac{5\sqrt{26}}{26};$$

Then

$$A(\alpha) = \frac{49.5 + 35.5\sqrt{26}}{26} = \frac{245 + 175\sqrt{26}}{26.}$$

Scores

Part I

Question	1	2	3	4	5	6	7
Points	1	1	1	1	1	1	1

Part 2

Question	1	2	3a	3b	4a	4b	4c	4d	5a	5b	6a	6b	7a	7b	7c
Points	1	1	1	1	1	0.5	0.5	0.5	1	1	0.5	1	1	1	1