Part 1

Select the correct option for each of the following questions. You do not need to present your calculations. Each correct answer is awarded 1 point. Each incorrect answer is penalized in 0,2 points.

1. Six students, three boys and three girls, lineup in a random order for a photograph. What is the probability that the boys and girls alternate?

(A) $^1/_{10}$ (B) $^1/_{36}$ (C) $^1/_5$ (D) $^1/_2$

2. There is a row in Pascal's Triangle where the fourth and fifth elements are equal. What is the sum of all elements in that row?

(A) 16   (B) 256   (C) 128   (D) 64

3. Consider two real numbers $a, b > 0$. Which of the following expressions is equivalent to $\ln(a^2 b) - \ln(b^2 a)$?

(A) $2(a + \ln b)$ (B) $2\ln(ab)$ (C) $\ln a - \ln b$ (D) $2\ln a + \ln b$

4. Consider the complex number $z = 8e^{i\frac{\pi}{6}}$. Which of the following complex numbers is a 6th-root of z?

(A) $\sqrt{2}e^{i\frac{25\pi}{36}}$ (B) $\sqrt{2}e^{-i\frac{\pi}{36}}$ (C) $2\sqrt{2}e^{i\frac{25\pi}{36}}$ (D) $2\sqrt{2}e^{-i\frac{\pi}{36}}$

5. For which of the following functions h is h', the derivative of h, positive for all $x \in \left] \frac{\pi}{2}, \pi \right[$?

(A) $h(x) = \sin(x) + \cos(x)$ (C) $h(x) = \cos(x) - \sin(x)$

(B) $h(x) = \sin(x) - \cos(x)$ (D) $h(x) = -\sin(x) - \cos(x)$

6. Consider the sequence defined by $u_n = n \ln(n+3) - n \ln(n)$.
The limit $\lim\limits_{n \to +\infty} u_n$ is

(A) 3   (B) 0   (C) e (D) e^3

7. Let f be a function with domain $\mathbb{R}^+$. The line of equation $y = \frac{1}{2}x + \frac{3}{2}$ is an asymptote to the graph of f. The value of $\lim\limits_{x \to +\infty} \dfrac{x + e^{\frac{1}{x}}}{f(x)}$ is

(A) 2   (B) 0   (C) $\frac{1}{2}$ (D) $+\infty$

Part 2

Provide a detailed justification for each one of your answers.

1. Consider the complex numbers

$$z_1 = -3 - \sqrt{3}i \quad \text{and} \quad z_2 = -\cos\left(-\frac{\pi}{3}\right) - i\sin\left(-\frac{\pi}{3}\right).$$

   (a) Write z_1 and z_2 in trigonometric form.

   (b) Determine the smallest value of $n \in \mathbb{N}$ such that $(z_1 z_2)^n$ is a positive real number.

2. Consider the real function defined by the expression

$$f(x) = x^2 + e^{x^2} - 1$$

Using exclusively analytic methods:

   (a) Study f regarding its monotony and local extrema.

   (b) Study function f regarding the regions where its graph is up or down concave.

   (c) Study the existence of asymptotes to the graph of f.

   (d) Show, using the intermediate value theorem, that there exists at least one $c \in]1, 2[$ such that $f(c) = 3c$.

   (e) Let g be the real function defined by $g(x) = \sin^2 x$.

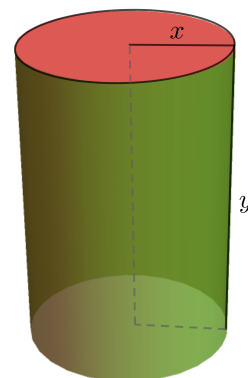   i) Determine the domain of the function $\frac{f}{g}$.

      ii) Compute $\lim\limits_{x \to 0} \frac{f}{g}(x)$.

3. The task at hand is to create a box with a circle-shaped base with radius $x\,cm$, and a height $y\,cm$ (see figure). The box must have a volume of $15\pi\,cm^3$. The side and bottom material (green) has a cost of 10 euros per cm^2 whereas the top material (pink) has a cost of 20 euros per cm^2

   (a) Show that, considering only the materials, the cost of producing the box is given by

$$C(x) = 30\pi x^2 + \frac{300}{x}$$

   (b) Determine the dimensions of the box so that the cost is as small as possible. Carefully justify your answer.

3

Scores:

Part I

Question	1	2	3	4	5	6	7
Score	1	1	1	1	1	1	1

Part 2

Question	1a	1b	2a	2b	2c	2d	2e	3a	3b
Score	2	1,5	1	1	1	1.5	1,5	2	1,5